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Geology. — *Preliminary Remarks on geological investigations in the Lesser Sunda islands near Australia.* By H. A. BROUWER.

(Communicated at the meeting of March 26, 1938.)

In 1937 collaborative researches were made by a group of geologists and advanced students of the University of Amsterdam on several of the Lesser Sunda islands near Australia. In the southern row of islands detailed investigations of critical parts were made in the island of Timor. In the northern row investigations were made on the islands Wetar, Alor, Lomblen, Adonara, Solor and in the eastern part of Flores. Some remarks on our preliminary results will be given in the following pages [1]).

Age of the Flysch in Timor.

The name Flysch was given by WANNER to a facies of rocks which resembles petrographically the Late Mesozoic and Eocene Flysch of the Alps and Carpathian Mountains. This formation is considered to belong to the upper part of the Triassic (Ladinic to Noric). A result of our investigations is that this facies comprises rocks of Upper-Paleozoic age.

On the overthrust sheets in Timor.

The structure of Netherlandic Timor has been discussed by former investigators, mainly by WANNER and MOLENGRAAFF.

For the tectonic relation between a lower unit with Flysch facies — of which the base has not been observed — and the superimposed rocks, sections in the central part of Netherlandic Timor, have proved to be particularly instructive.

The folded lower unit is separated by an intensely crushed and squeezed zone from overlying Permian marls, limestones, tuffs and effusive rocks of which the facies is very different from that of the Permian of the lower unit. It must have moved horizontally over large distances. The outcrop of the contact between the lower unit and the superimposed rocks is strongly influenced by axial dip. Judging from the distribution of the different rocks in the northern part of Netherlandic Timor there may be several windows of the lower unit, which are surrounded by regions with crystalline schists and many varying Paleozoic, Mesozoic and Tertiary formations, in some of which a great wealth of fossils is found.

Our results must be worked out in detail before a division in structural units of the rocks on top of the lower unit can definitively be given.

Comparison between the two rows of islands.

The two rows contrast strongly with regard to their geological composition and structure. A great part of the northern row is covered by volcanics and many volcanoes are still active. Intrusive granitic to dioritic rocks and various sediments are also found. Although rocks from this row have been mentioned as pre-Tertiary in the literature, we have not discovered rocks older than Young-Tertiary in age and we only saw rather simple structures at all those places where our investigations were made.

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[1]) As a matter of course many of these preliminary remarks are based upon the detailed work of my collaborators, who will publish the results of their work in due time in special publications.

The southern row is entirely devoid of active volcanoes and Paleozoic, Mesozoic and Tertiary rocks with a very complicated structure are widely distributed.

Besides the strong contrast in geological composition and structure between the two rows, there are points of resemblance. The volcanic rocks, which occur near the north coast of Timor, resemble the older part of the volcanic rocks of the northern row. We could say that rocks of the northern row rest upon rocks of the southern row near the north coast of Timor, while nowhere in the northern row the base of these younger rocks has been found uncovered by erosion.

Tectonic evolution.

There are Old-Tertiary and Younger-Mesozoic 1) elements in a complex of mostly dark coloured sediments as greywackes, conglomerates, sandstones, clay shales, cherts and limestones, which is found in close relation with the crystalline schists of Timor. Different horizons are found transgressive on the schists. These Younger-Mesozoic and Old-Tertiary formations indicate periods of diastrophism in Mesozoic time.

The present structure is the end result of long periods of movement and the question arises how much of the horizontal movement which is proved by the overthrusting took place before Tertiary times. We will only mention in this connection that the intensity of folding of the transgressive Younger-Mesozoic and Old-Tertiary formations is varying for different sections and at different horizons in the same section. At some places slightly folded or even nearly horizontal rocks are found. We will not enter now farther into the details of this question.

With regard to the youngest tectonic evolution of the rows of islands additional facts have been found, which agree with our opinion expressed in former publications, that in this region there is to be found a fine illustration of a simple relationship between crustal movements and volcanic activity, the volcanic action having been more prolonged in the northern row in proportion as the island rows are further separated.

Proofs of the young tectonic activity in both rows are amply given by the upheaved terraces and coral reefs and the fractures and faults near the surfaces of the present islands.

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1) Upper-Jurassic (Oxfordian) is represented according to Professor R. HEINZ in Leipzig, who made preliminary determinations of our species of *Inoceramus.*

Embryology. — *Inducing capacity of the embryonic eye.* By M. W. WOERDEMAN.

(Communicated at the meeting of March 26, 1938.)

By far the most important question occupying experimenting embryologists is the problem of embryonic induction.

By this is meant an influence of certain cell groups in the embryonic body on not yet differentiated cells, as the result of which a special tissue or organ is formed out of the latter cells.

The roof of the archenteron, for example, exerts an inducing action on that part of the ectoderm with which it is in contact, causing this ectodermal region to form the neural plate (anlage of the nervous system) (SPEMANN and pupils).

It appeared that such an organ, formed by inducing action, as the neural plate, in its turn can also exert inducing actions, for MANGOLD (1928) pointed out that fragments of neural plate and even fragments of brain tissue, introduced under indifferent ectoderm, may induce part of a neural plate in that ectoderm.

It is further a well-known fact that the optic vesicles, formed as evaginations of the neural tube, where they get into contact with the ectoderm of the head induce a lens in that ectoderm, and consequently also possess an inducing capacity.

Recently many investigators are engaged on the question how the inducing action is brought about. It has been suggested that it might be due to production of chemical substances by the inductors (inductive substances). Although many consider this an established fact, yet, on deeper contemplation of the induction phenomena, there are in my opinion various difficulties which cannot be immediately solved by the hypothesis of a chemical inducing action. But even if this hypothesis should be accepted, still many questions remain to be answered.

On a previous occasion I made a communication on this subject (WOERDEMAN, 1936).

One of the numerous questions arising in the study of embryonic induction will now be discussed more in detail.

Although the optic vesicles are evaginations of the neural tube, they induce lenses in the ectoderm, while fragments of the neural tube itself and even parts of the brain developed from the neural tube induce the formation of neural plates.

Have we to deal here with different inducing actions or is the ectoderm, on which the optic vesicle in normal development exerts its action and which has had part in gastrulation and neurulation, different from the ectoderm out of which a neural plate may be induced?

In other words, must we assume either different, for each developmental process specific, inducing actions (c.q. inductive substances) or a gradual modification of the reacting embryonic material, owing to which fact the reaction on the same stimulus in younger stages of development would be different from that in older stages?

Or is a specific inducing action as well as a special phase of development of the reacting cell material responsible for the development of a special organ?

Various investigations are already known, from which conclusions may be drawn to answer these questions.

I myself made a number of experiments which may contribute to the elucidation of this problem.

The idea on which this research was based was bringing optic vesicles of different ages into contact with ectoderm of different stages of development.

This might be done by removing the ectoderm of the head which covers the optic vesicle and replacing it by ectoderm taken from another region of the same or of another embryo.

In this way I made numerous experiments in 1936, but I found that the explanation of the results offered difficulties, since the grafted ectoderm was not only influenced by an inducing action of the optic vesicle but probably also by that of other adjacent organs of the head. This action was also met with by MANGOLD (1933) and HOLTFRETER (1933) in their experiments. For this reason I have afterwards applied two other methods for my research.

In the first place extirpated optic vesicles were brought in the blastocœl of young gastrulae, which enabled us to examine their influence on the young ectoderm.

In the second place I grafted optic vesicles in the abdominal wall and then covered them with ectoderm of different origin and age. In this series of experiments I also used presumptive ectoderm from young gastrulae, since it is not impossible that the presumptive ectoderm during the process of gastrulation is changed and consequently by the first method of operating the eyes introduced into the blastocœl will come into contact with ectoderm which has been modified during gastrulation. If presumptive ectoderm of a stage previous to gastrulation is grafted on the optic vesicle in an old host, it will be possible to study the reaction of ectoderm which has not gone through gastrulation and will no longer get the opportunity to have part in it, while introduction of eyes into the blastocœl shows us the behaviour of ectoderm which takes part in the gastrulation. In the same way the influence of neurulation on the ectoderm may be investigated by grafting on the optic vesicle implanted into an old host presumptive ectoderm of donors before and after neurulation.

One might wonder why the explantation method has not been applied in order to prevent that the host in which the eyes were introduced influences the process (see WOERDEMAN, 1936).

The reason may be found in my supposition that the medium in which the explants are mostly cultivated is not completely indifferent and certainly not physiological. Grafting in another embryo offers the best chance to the implants for a good development. However, I have tried to graft the implants as far away from inducing organs as possible (ventral side of the embryo) and at the same time taken care that the grafts were made in larvae which sometimes had the same age as the donor of the optic vesicles and of the ectoderm and at other times differed considerably in age from them, in order to avoid neglecting a possible influence of the host. By application of an accurately aseptic operation technique (WOERDEMAN, 1930) all experimental animals were kept alive and I may now report on 70 experiments, most of which were performed on embryos of *Axolotl*. A smaller number of experiments were made on larvae of *Triton taeniatus*.

Of the many remarkable phenomena which became manifest on microscopical examination of the experimental larvae only a few will be mentioned here.

In the first place, not in any case I could with certainty demonstrate the induction of neural tissue by the optic vesicle, although the ectoderm with which it was brought into contact certainly in several cases possessed the potency to form neural tissue.

This result corroborates similar observations by G. A. SCHMIDT and M. N. RAGOSINA (1936, 1937), made on introduction of optic vesicles of *Bombinator, Rana* and *Pelobates* into blastulae of *Triton*.

The Russian investigators, however, communicate that they were able to obtain neural induction, if together with the implanted optic vesicle a fragment of neural tissue had been introduced.

In *Triton* as host I also noticed some cases where it was not improbable that a fragment of neural tissue had been induced. In such cases a small part of the neural tube had been introduced into the blastocœl together with the eye.

In *Axolotl*, however, I could not in any case observe neural induction, not even in embryos in which, together with the eye, a fragment of the neural tube had been grafted and the neural tissue, developed from it, had come into contact with the ectoderm.

Further it appeared that no part is played here by the developmental processes which the ectoderm has previously gone through, for ectoderm which had taken part in gastrulation as well as ectoderm which had no part in this process reacted in the same way on the inducing action of optic vesicle and neural material which might have been grafted at the same time.

In the second place, as was to be expected, the ectoderm reacted on the inducing action of the eye with the formation of lenses.

It became evident that also ectoderm which had not gone through gastrulation and neurulation could form lens vesicles, which is a corroboration of WADDINGTON's discovery (1936) with *Triton alpestris* and that of FILATOW (1937) with *Rana temporaria*.

In table I may be seen in which combinations of eye (Inductor) and

Table I.

Axolotl	Reactive System	Stage 10	Stage 11-12	Stage 14	Stage 15-17	Stage 19	Stage 20-21	Stage 22-23	Stage 24	Stage 25
Inductor										
Stage	20				⊕		⊕⊕			
»	21				⊕⊕⊕⊕		⊕⊕		⊕	
»	22							⊕		
»	24								⊕	
»	24-25	⊕⊕⊕⊕ ⊕⊕⊕⊕⊕								
»	25	⊕								⊕
»	27				⊕⊕⊕					
»	27-28						⊕			
»	28	⊕⊕								
»	28-29									
»	30-31		⊕							
»	31									
»	32							⊕		
»	34				⊕⊕					

⊕ Lens

ectoderm (Reactive system) lens formation has been observed. The numbering of the stages is that given by HARRISON for *Amblystoma punctatum*.

Since opinions differ on this subject (see for older researches: MANGOLD, 1931, and further of recent times: BRÖER and v. UBISCH, 1934; MANUYLOWA and KISLOW, 1934; PERRI, 1934 and 1936; GOSTEJEWA, 1935; POPOV, 1936 and 1937) it may be communicated that in my experimental object, *Axolotl*, in the examined stages still the whole ectoderm (head, dorsal, lateral and ventral surface of the trunk) is able to react with lens formation on the inducing action of the eye.

Further particulars concerning lens induction cannot be discussed here.

While in my opinion from the observation that ectoderm competent to form neural plate does not react on the inducing action of the eye with the formation of neural plate but with lens formation, it may already be concluded that there is a certain specificity in the induction, this is still more clearly proved by further observations.

In 24 cases a nasal pit appeared to be induced in my *Axolotl* material, while also in *Triton* this phenomenon was repeatedly observed. In this communication, however, I shall restrict myself to the examination of *Axolotl*.

From table II it appears that particularly inductors of younger stages possess the capacity to induce nasal pits.

Table II.

Axolotl / Reactive System	Stage 10	Stage 11-12	Stage 14	Stage 15-17	Stage 19	Stage 20-21	Stage 22-23	Stage 24	Stage 25
Inductor									
Stage 20				•		••••			
» 21				•••		•		•	
» 24								•••	
» 24-25	••••••							•••	
» 25							•		
» 27	σ								
» 27-28						•			
» 28	σσσσ								
» 28-29									
» 30-31			σσ						
» 31				σσσσ					
» 32									

σ ear vesicle
• nasal pit

Here also young as well as older ectoderm reacts and at the same time it became apparent that a nasal pit may be formed not only out of ectoderm of the head but from the whole ectoderm.

Examining the literature, we find that usually the prosencephalon is considered responsible for the induction of nasal pits, since various investigators in induction experiments observed the formation of nasal pits in the neighbourhood of prosencephalon tissue (MANGOLD, 1933;

RAVEN, 1933 and 1935; HOLTFRETER, 1936), while ZWILLING (1934) with special experiments produced arguments for this conception. On the ground of debatable experiments SIGGIA (1936) arrives at the conclusion that the roof of the archenteron plays a part in the induction, while HOLTFRETER does not exclude the possibility that in addition to the prosencephalon as most important inductor there are still other inductors for the nasal pit.

I have now paid special attention to the fact whether with the grafted eyes which had induced nasal pits also neural tissue had been implanted. Indeed, in all cases I could point out the presence of (sometimes very small) quantities of brain tissue by the side of the eyes and connected with them (so probably removed and grafted together).

Consequently I must conclude from my preparations that the induction of nasal organs takes place by that part of the neural tube which lies in the immediate surroundings of the optic evagination and in normal development is in contact with the ectodermal placode, out of which the nasal pits are formed (KNOUFF, 1935).

It does not seem likely to me that the optic vesicle itself induces the nasal organs, although I have arguments for the opinion that it is of importance to the further differentiation of the once induced nasal pit.

For the problem on which we are engaged now it is important to be able to state that part of the neural tissue on ectoderm, which certainly has the "competence" to react with a formation of neural plate (stages 10—12), does not exert a neural inducing action but induces nasal pits.

This phenomenon also pleads for a specificity of the inducing capacity. I am the more strongly inclined to think that this conception is correct after examination of cases, as I observed them repeatedly, where closely by the side of each other from the same ectoderm a lens was formed, lying in the pupil of a grafted eye, and a nasal pit, situated against a fragment of brain tissue, which was connected with the eye and evidently had been grafted together with it.

Attention should be drawn also to a remarkable fact which may be observed in Table II.

Optic vesicles, from stages older than stage 25, evidently do not induce any more nasal pits in ectoderm which reacts on the inducing action of younger inductors with the formation of nasal organs.

In 11 cases, however, I observed that these older inductors, besides lenses, also had induced ear vesicles.

They were formed not only out of ectoderm of the head (in that case it might be supposed that the implant had increased an already present latent potency to form ear vesicles; IKEDA, 1937), but also out of ventral ectoderm and presumptive buccal ectoderm. The inductors were sometimes accompanied by a quantity of brain tissue, but in half the cases brain tissue which had been grafted at the same time was completely wanting and the inductor consisted only of an eye cup.

It was remarkable that the ear vesicles as a rule were situated where the nerve fibres of the retinal layer of the inducing eye cup leave its chorioid fissure and grow into the surroundings.

As far as my observations at present have proceeded I am compelled to assume that the optic vesicle itself may induce the ear vesicles.

This is certainly remarkable, since it is not yet known for certain how the ear vesicle is induced.

SZEPSENWOL (1933) considers the acousticofacial ganglion the inductor, DALCQ (1933) looks for the primary inductor in the entomesoderm and assigns a secondary role to the rhombencephalon (particularly the neural crest), while the histological differentiation would be dependent on the presence of the acousticofacial ganglion.

MANGOLD (1937) comes to the same conclusions for the Urodela as his collaborator GUARESCHI (1935) for the Anura, viz. that the neural fold is the inductor of the ear vesicle. The question whether the neural part or the mesectodermal portion induces had to be left undecided. According to him, a role of the entomesoderm should not be excluded.

HOLTFRETER (1936) enumerates various possibilities, mentioning the neural crest as "potential inductor".

The eye, however, is not mentioned at all.

Further experiments will have to decide here.

For the sake of completeness it should be mentioned that twice I found a ganglion by the side of an ear vesicle. In the remaining nine cases there were no traces of ganglia. Opinions differ also on the manner in which the acoustic ganglion is formed. In the two above-mentioned cases the ganglion may also have been induced, or developed from the induced ear vesicle or grafted together with the inductor. That cannot be decided afterwards from the preparations.

The fact that the same ectoderm may react simultaneously on the action of a special inductor (cf. tables I and II) with the formation of a lens and an ear vesicle is, in my opinion, again a strong indication that the inducing action cannot be explained simply by assuming a single inductive substance.

It seems to me that there are no reliable grounds to accept the incidentally mentioned supposition of MANGOLD (1936*a* and *b*) that different inductions might be due to different concentrations of one and the same inductive substance.

On the ground of my experiments I am more and more inclined to think that for the explanation of the inducing actions a certain specificity should be taken into consideration.

From the tables I and II it may be seen that within the stages I examined for induction of lenses and nasal pits no distinct influence of the age of the reactive system becomes manifest. Yet, in the literature there are several observations which compel us to take this into account as well. Perhaps also my observation that in ectoderm of stages older than stage 17 so far no induction of ear vesicles could be perceived, may be of importance.

Finally it may be mentioned that my assistant, van DETH, succeeded in studying the inducing action of the optic vesicle by means of the explantation method with chicken embryos. By cultivating ectoderm from chicken embryos in vitro and bringing the cultures into contact with optic vesicles, likewise from chicken embryos, van DETH could cause lens induction in vitro.

This method will, we hope, enable us to study the problems discussed here in future not only with material from amphibians but also on warm-blooded animals. Experiments with mammals are in preparation.

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Mathematics. — *Sur deux, trois ou quatre nombres premiers.* Par J. G. VAN DER CORPUT. (Cinquième communication.)

(Communicated at the meeting of March 26, 1938.)

Traitons d'abord quelques cas spéciaux de la proposition 3; dans ces cas spéciaux (les propositions 4, 5, 6 et 7) γ désigne un nombre positif, m un nombre naturel, N un nombre > 3, s un entier qui est en valeur absolue inférieur à $\gamma N^g n^\gamma$ où $n = \log N$, et $\psi(x)$ un polynôme du degré précis $g \geqq 1$ qui prend une valeur entière pour toute valeur entière de x; les nombres $c_{46}, c_{47}, \ldots, c_{51}$, figurant dans ces cas spéciaux, sont des nombres convenablement choisis, dépendant uniquement de γ, m et du choix du polynôme $\psi(x)$.

Proposition 4: *Désignons par $F(t)$ le nombre des manières dont on peut écrire t comme la somme de deux nombres premiers > 3 et posons*

$$\Phi(t) = \underset{\substack{u \geqq 3 \quad u' \geqq 3 \\ |u+u'-t| \leqq \frac{1}{2}}}{\int \int} \frac{du\, du'}{\log u \log u'}.$$

Si $\psi(x) + s$ est pair pour tout entier x, on a

$$\sum_{x=1}^{[N]} \left| F(\psi(x)+s) - C' \Phi(\psi(x)+s) \Pi \frac{w-1}{w-2} \right| < c_{46} N^{g+1} n^{-m};$$

le produit $\Pi \frac{w-1}{w-2}$ est étendu à tous les facteurs premiers impairs de $\psi(x)+s$, tandis que C' est la constante absolue

$$C' = 2 \prod_{\substack{w > 2 \\ w \text{ premier}}} \left(1 - \frac{1}{(w-1)^2}\right).$$

Démonstration: Pour tout nombre naturel $x \leqq N$ on a $|\psi(x)| \leqq c_{47} N^g$, donc

$$|\psi(x) + s| \leqq (c_{47} + \gamma) N^g n^\gamma.$$

Posons

$$A = A' = 3; \quad B = B' = \tfrac{1}{4} Z^g = (c_{47} + \gamma) N^g n^\gamma + \tfrac{1}{2}, \quad Z > 0.$$

$F(\psi(x)+s)$ est le nombre des manières d'écrire $\psi(x)+s$ sous la forme $p+p'$, où les nombres premiers p et p' sont supérieurs à 3 et inférieurs à $\psi(x)+s$, donc inférieurs à $(c_{42}+\gamma)\,N^g n^\gamma$, d'où il suit $A<p<B$ et $A'<p'<B'$. En outre

$$\Phi(\psi(x)+s)=\underset{|u+u'-\psi(x)-s)\leqq\frac{1}{2}}{\int_A^B\int_{A'}^{B'}}\frac{du\,du'}{\log u\log u'},$$

car les inégalités

$$u+u'\leqq\psi(x)+s+\tfrac{1}{2}\leqq(c_{47}+\gamma)\,N^g n^\gamma+\tfrac{1}{2}=B=B'$$

entraînent $u\leqq B$ et $u'\leqq B'$. En vertu de $Z>N$ la proposition 3, appliquée avec $m+1+\left[\gamma+\frac{\gamma}{g}\right]$ au lieu de m, nous donne par conséquent

$$\sum_{x=1}^{[N]}\left|F(\psi(x)+s)-\Phi(\psi(x)+s)\sum_{q=1}^{\infty}H(q,\psi(x)+s)\right|<c_{48}\,N^{g+1}\,n^{-m}.$$

En outre

$$\sum_{q=1}^{\infty}H(q,t)=2\prod_{w>2}(1+H(w,t))$$

est d'après le lemme 18 pour tout nombre pair t égal à

$$2\prod_{\substack{w>2\\w|t}}\left(1+\frac{1}{w-1}\right)\cdot\prod_{\substack{w>2\\w\nmid t}}\left(1-\frac{1}{(w-1)^2}\right)=C'\prod_{\substack{w>2\\w|t}}\frac{1+\dfrac{1}{w-1}}{1-\dfrac{1}{(w-1)^2}},$$

d'où suit l'assertion en vertu de l'identité

$$\frac{1+\dfrac{1}{w-1}}{1-\dfrac{1}{(w-1)^2}}=\frac{w-1}{w-2}.$$

En utilisant les lemmes 18, 19 et 20 il est facile de déduire les propositions 5, 6 et 7 de la proposition 3.

Proposition 5: *Soit $3\leqq A<B$; désignons par $F(t)$ le nombre des manières dont on peut écrire t sous la forme $p+p'$, où les nombres premiers p et p' sont situés entre A et B; posons*

$$\Phi(t)=\underset{|u+u'-t|\leqq\frac{1}{2}}{\int_A^B\int_A^B}\frac{du\,du'}{\log u\log u'}.$$

Si $\psi(x)+s$ est pair pour tout entier x, on a pour tout nombre $Z\geqq\sqrt[g]{4B}$

$$\sum_{|x|\leqq Z}\left|F(\psi(x)+s)-C'\,\Phi(\psi(x)+s)\prod\frac{w-1}{w-2}\right|<c_{49}\,Z^{g+1}\,z^{-m};$$

C' et $\Pi \dfrac{w-1}{w-2}$ sont la constante absolue et le produit, figurant dans la proposition précédente.

Proposition 6: *Désignons par $F(t)$ le nombre des manières dont on peut écrire t sous la forme $p_1^2 + p_2^2 + p'$, où p_1, p_2 et p' désignent des nombres premiers > 3; posons*

$$\Phi(t) = \underset{\substack{u_1 \geqq 3 \quad u_2 \geqq 3 \quad u' \geqq 3 \\ |u_1^2 + u_2^2 + u' - t| \leqq \frac{1}{2}}}{\int \int \int} \frac{du_1\, du_2\, du'}{\log u_1 \log u_2 \log u'}.$$

Si $\psi(x)$ est $\equiv 1$ ou $\equiv 3$ (mod. 6) pour tout entier x, on a

$$\sum_{x=1}^{[N]} |F(\psi(x)+s) - 2\,\Phi(\psi(x)+s) \underset{\substack{w>2 \\ w \text{ premier}}}{\Pi} (1 + H(w, \psi(x)+s))| < c_{50} N^{g+1} n^{-m};$$

pour tout nombre premier $w > 2$ et pour tout entier $t \equiv 1$ ou $\equiv 3$ (mod. 6), on a

$$H(w, t) = -\frac{w+1}{(w-1)^2}, \text{ lorsque } w \equiv 1 \text{ (mod. 4) et } w \text{ est un facteur de } t;$$

$$= \frac{1}{w-1}, \text{ lorsque } w \equiv -1 \text{ (mod. 4) et } w \text{ est un facteur de } t;$$

$$= \frac{3w+1}{(w-1)^3}, \text{ lorsque } w \equiv 1 \text{ (mod. 4) et } t \text{ est un reste quadratique de } w;$$

$$= -\frac{1}{(w-1)^2}, \text{ lorsque } w \equiv 1 \text{ (mod. 4) et } t \text{ est un non-reste de } w;$$

$$= \frac{w+1}{(w-1)^3}, \text{ lorsque } w \equiv -1 \text{ (mod. 4) et } t \text{ est un reste quadratique de } w;$$

$$= -\frac{3w-1}{(w-1)^3}, \text{ lorsque } w \equiv -1 \text{ (mod. 4) et } t \text{ est un non-reste de } w.$$

Proposition 7: *Désignons par $F(t)$ le nombre des manières dont il est possible d'écrire t comme la somme des carrés de quatre nombres premiers > 3, et posons*

$$\Phi(t) = \underset{\substack{u_1 \geqq 3 \quad u_2 \geqq 3 \quad u_1' \geqq 3 \quad u_2' \geqq 3 \\ |u_1^2 + u_2^2 + u_1'^2 + u_2'^2 - t| \leqq \frac{1}{2}}}{\int \int \int \int} \frac{du_1\, du_2\, du_1'\, du_2'}{\log u_1 \log u_2 \log u_1' \log u_2'}.$$

Si $\psi(x) + s \equiv 4$ (mod. 24) pour tout entier x, on a

$$\sum_{x=1}^{[N]} |F(\psi(x)+s) - 8\,\Phi(\psi(x)+s) \underset{\substack{w \text{ premier} \\ w>2}}{\Pi} (1 + H(w, \psi(x)+s)| < c_{51} N^{g+1} n^{-m};$$

on a pour tout nombre premier $w > 2$ *et pour tout nombre* $t \equiv 4$ (*mod.* 24)

$$\begin{aligned}
H(w,t) &= \frac{w^2+6w+1}{(w-1)^3}, \text{ lorsque } w \equiv 1 \text{ (mod. 4) et } w \text{ est un facteur de } t; \\
&= \frac{w^2-6w+1}{(w-1)^3}, \text{ lorsque } w \equiv -1 \text{ (mod. 4) et } w \text{ est un facteur de } t; \\
&= -\frac{5w^2+10w+1}{(w-1)^4}, \text{ lorsque } w \equiv 1 \text{ (mod. 4) et } t \text{ est un reste quadratique de } w; \\
&= \frac{3w+1}{(w-1)^3}, \text{ lorsque } w \equiv 1 \text{ (mod. 4) et } t \text{ est un non-reste de } w; \\
&= \frac{(w+1)(3w-1)}{(w-1)^4}, \text{ lorsque } w \equiv -1 \text{ (mod. 4) et } t \text{ est un reste quadratique de } w; \\
&= -\frac{5w^2-10w+1}{(w-1)^4}, \text{ lorsque } w \equiv -1 \text{ (mod. 4) et } t \text{ est un non-reste de } w.
\end{aligned}$$

Lemme 25: *Les fonctions* $H(q,t)$, *définies par* (35), (36) *et* (37) *satisfont pour tout entier* $t \neq 0$ *qui satisfait à la congruence correspondante, à l'inégalité*

$$\sum_{q=1}^{\infty} H(q,t) > \frac{c_{52}}{\log(3+|t|)}; \quad \ldots\ldots\ldots \quad (56)$$

dans ce lemme $c_{52}, c_{53}, \ldots, c_{58}$ *sont des constantes absolues et positives, convenablement choisies.*

Démonstration: D'après la remarque, ajoutée à la proposition 3, on a

$$\sum_{q=1}^{\infty} H(q,t) = \varrho \prod_{\substack{w \text{ premier} \\ w>2}} (1 + H(w,t)),$$

où $\varrho = 2$ ou 8 et $1 + H(w,t) > 0$.

Pour tout nombre premier $w > 2$, qui n'est pas un facteur de t, on a, d'après les lemmes 18, 19 et 20,

$$1 + H(w,t) > e^{-\frac{c_{53}}{w^2}},$$

de sorte que le produit $\Pi\,(1 + H(w,t))$, étendu à ces nombres premiers, est supérieur à c_{54}.

Pour tout facteur premier impair w de t on a, d'après les mêmes lemmes,

$$1 + H(w,t) > e^{-\frac{1}{w}-\frac{c_{55}}{w^2}}$$

de sorte que le produit $\Pi\ (1 + H(w, t))$, étendu à ces nombres premiers, est supérieur à

$$e^{-c_{56} - \sum\limits_{w \leqq |t|} \frac{1}{w}} > e^{-c_{57} - \log\log(3+|t|)} = \frac{c_{58}}{\log(3+|t|)}.$$

Le membre de gauche de (56) est donc supérieur à $\dfrac{c_{54}\, c_{58}}{\log(3+|t|)}$.

Finissons cet article en démontrant la proposition 1 (première communication, p. 847). Dans cette démonstration $c_{59}, c_{60}, \ldots, c_{69}$ désignent des nombres positifs, convenablement choisis et dépendant uniquement de γ, m et du choix du polynôme $\psi(x)$.

Pour tout nombre x qui est en valeur absolue $\leqq Z$, nous avons

$$|\psi(x)| \leqq c_{59}\, Z^g, \text{ donc } |\psi(x) + s| \leqq (c_{59} + \gamma)\, Z^g\, z^\gamma.$$

Posons

$$A = A' = A_1 = A_2 = A_1' = A_2' = 3;$$
$$B = B' = B_1^2 = B_2^2 = B_1'^2 = B_2'^2 = \tfrac{1}{4} N^g = 4\,(c_{59} + \gamma)\, Z^g\, z^\gamma.$$

La proposition 3, appliquée avec N au lieu de Z et avec $m + 1 + \left[\gamma + \frac{\gamma}{g}\right]$ au lieu de m, nous apprend en vertu de $N > Z$

$$\sum_{|x| \leqq Z}{}' \left|F(\psi(x) + s) - \Phi(\psi(x) + s) \sum_{q=1}^{\infty} H(q, \psi(x) + s)\right| < c_{60}\, Z^{g+1-m}\, z. \quad (57)$$

Examinons d'abord les expressions $f_1 = p + p'$; $f_2 = p_1^{\,2} + p_2^{\,2} + p'$ et $f_3 = p_1^{\,2} + p_2^{\,2} + p_1'^{\,2} + p_2'^{\,2}$.

Pour chacune de ces expressions, la fonction $\Phi(t)$, définie dans (32), (33) ou (34), jouit de la propriété

$$\Phi(t) > c_{61} \frac{t}{\log^4 t} \qquad (1 + 4 \,.\, 3^4 \leqq t \leqq \tfrac{1}{4} B). \quad . \quad . \quad . \quad (58)$$

Désignons par A le nombre des entiers positifs $x \leqq Z$ tels que

$$\psi(x) + s \geqq Z^g\, z^{-\frac{1}{2} m} \text{ et } F(\psi(x) + s) = 0.$$

Il suit de (56) et (58) que tous ces nombres x satisfont à l'inégalité

$$\Phi(\psi(x) + s) \sum_{q=1}^{\infty} H(q, \psi(x) + s) > c_{62}\, Z^g\, z^{-\frac{1}{2} m - 5},$$

de sorte que (57) nous apprend

$$c_{62}\, A\, Z^g\, z^{-\frac{1}{2} m - 5} < c_{60}\, Z^{g+1}\, z^{-m},$$

donc

$$A < c_{63}\, Z\, z^{-\frac{1}{2} m + 5}.$$

Le nombre des entiers positifs tels que $\psi(x)+s<Z^g z^{-\frac{1}{2}m}$ est inférieur à $c_{64} Z z^{-\frac{m}{2g}}$. Par conséquent le nombre des entiers positifs $\leqq Z$ tels que $F(\psi(x)+s)=0$, est inférieur à $c_{63} Z z^{-\frac{1}{2}m+5}+c_{64} Z z^{-\frac{m}{2g}}$.

Ceci vaut pour tout nombre naturel m, d'où il suit que le nombre des entiers positifs $\leqq Z$ tels que $F(\psi(s)+s)=0$ est inférieur à $c_{65} Z z^{-m}$.

Ainsi la proposition 1 est démontrée pour les expressions f_1, f_2 et f_3.

Examinons finalement les expressions

$$f_4=p-p'; \quad f_5=-p_1^2-p_2^2+p'; \quad f_6=p_1^2-p_2^2+p';$$
$$f_7=p_1^2+p_2^2+p_1'^2-p_2'^2 \quad \text{et} \quad f_8=p_1^2+p_2^2-p_1'^2-p_2'^2.$$

Maintenant nous avons

$$\Phi(t)>c_{66} Z^g z^{-4} \qquad (|t| \leqq \tfrac{1}{4} B),$$

donc en vertu de (56)

$$\Phi(t) \sum_{q=1}^{\infty} H(q,t)>c_{67} Z^g z^{-5} \qquad (|t| \leqq \tfrac{1}{4} B).$$

De (57) il résulte par conséquent que le nombre des entiers x tels que $|x| \leqq Z$, $\psi(x)+s \neq 0$ et $F(\psi(x)+s)=0$, est inférieur à $c_{68} Z z^{-m+5}$.

Ceci vaut pour tout nombre naturel m et il y a tout au plus g nombres x tels que $\psi(x)+s=0$. Le nombre des entiers x tels que $|x| \leqq Z$ et $F(\psi(x)+s)=0$ est donc inférieur à $c_{69} Z z^{-m}$, de sorte que la proposition 1 est complètement démontré.

Mathematics. — *Contribution à la théorie additive des nombres.* Par J. G. VAN DER CORPUT. (Deuxième communication).

(Communicated at the meeting of March 26, 1938.)

Troisième partie de la démonstration de la proposition 1.

Il nous reste seulement de démontrer que la contribution U_6 au membre de droite de (13) des points α, β tels que

$$\left| \alpha - \frac{a}{q} \right| < N^{-1} n^{\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

a un ordre de grandeur tout au plus égal à celui de $\Gamma^2 \Gamma'^2 N^3 n^{-m}$. En effet, en changeant α et β, $\frac{a}{q}$ et $\frac{a'}{q'}$, on trouve alors que la contribution des points α, β tels que

$$\left| \beta - \frac{a'}{q'} \right| < N^{-1} n^{\tau},$$

donc, a fortiori, celle des points α, β satisfaisant au système des inégalités

$$\left| \beta - \frac{a'}{q'} \right| < N^{-1} n^{\tau}, \qquad \left| \alpha - \frac{a'}{q'} \right| \geqq N^{-1} n^{\tau},$$

possède un ordre de grandeur tout au plus égal à celui de $\Gamma^2 \Gamma'^2 N^3 n^{-m}$.

Dans chaque intervalle $j\left(\frac{a}{q}\right)$ on a

$$\left| F_{\alpha}\left(\frac{a}{q}\right) - \Phi_{\alpha}\left(\frac{a}{q}\right) \right| = \left| \sum_{v} r(v)\, e^{2\pi i \alpha v} - \lambda\left(\frac{a}{q}\right) \sum_{v} \varrho(v)\, e^{2\pi i \left(\alpha - \frac{a}{q}\right) v} \right|.$$

L'inégalité (4), appliquée avec $m + 1 + (4l + 4)\sigma + 2\tau$ au lieu de m, nous apprend pour toute valeur réelle de y

$$\left| \sum_{v \leqq y} r(v)\, e^{\frac{2\pi i a v}{q}} - \lambda\left(\frac{a}{q}\right) \sum_{v \leqq y} \varrho(v) \right| \leqq c_9\, \Gamma N q^l\, n^{-m-1-(4l+4)\sigma-2\tau},$$

d'où il suit en vertu du lemme 2

$$\left| F_{\alpha}\left(\frac{a}{q}\right) - \Phi_{\alpha}\left(\frac{a}{q}\right) \right| \leqq 3 c_9\, \Gamma N q^l\, n^{-m-1-(4l+4)\sigma-2\tau} \left\{ 1 + N \left| \sin \pi \left(\alpha - \frac{a}{q} \right) \right| \right\}$$

$$< c_{10}\, \Gamma N n^{-m-1-(3l+4)\sigma-\tau}$$

pour les nombres α satisfaisant à (17).

On obtient d'une manière analogue

$$\left| F'_\alpha\left(\frac{a}{q}\right) - \Phi'_\alpha\left(\frac{a}{q}\right) \right| < c_{11}\, \Gamma'\, N n^{-m-1-(3l+4)\sigma-\tau}.$$

La relation (10) étant vérifiée, on a

$$\left| \Phi_\alpha\left(\frac{a}{q}\right) \right| \leqq \gamma_1\, q^l \sum_v |\varrho(v)| \leqq \gamma_1\, \Gamma N n^{l\sigma};$$

$\varrho'(v')$ étant en valeur absolue $\leqq \Gamma'$, on obtient en outre

$$\left| \Phi'_\alpha\left(\frac{a}{q}\right) \right| \leqq \gamma_1\, q^l \sum_{v'} |\varrho'(v')| \leqq \gamma_1\, \Gamma' N n^{l\sigma}.$$

De cette manière on trouve que

$$\begin{aligned} G_\alpha\left(\frac{a}{q}\right) &= F_\alpha\left(\frac{a}{q}\right) F'_\alpha\left(\frac{a}{q}\right) - \Phi_\alpha\left(\frac{a}{q}\right) \Phi'_\alpha\left(\frac{a}{q}\right) \\ &= \left(F_\alpha\left(\frac{a}{q}\right) - \Phi_\alpha\left(\frac{a}{q}\right)\right) \Phi'_\alpha\left(\frac{a}{q}\right) \\ &\quad + \left(F'_\alpha\left(\frac{a}{q}\right) - \Phi'_\alpha\left(\frac{a}{q}\right)\right) \Phi_\alpha\left(\frac{a}{q}\right) \\ &\quad + \left(F_\alpha\left(\frac{a}{q}\right) - \Phi_\alpha\left(\frac{a}{q}\right)\right)\left(F'_\alpha\left(\frac{a}{q}\right) - \Phi'_\alpha\left(\frac{a}{q}\right)\right) \end{aligned}$$

est en valeur absolue inférieure à $c_{12}\, \Gamma \Gamma'\, N^2\, n^{-m-1-(2l+4)\sigma-\tau}$. Les relations (10) et (2) nous apprennent, puisque $\varrho'(v')$ est en valeur absolue $\leqq \Gamma'$,

$$\left| F_\beta\left(\frac{a'}{q'}\right) \right| \leqq \sum_v |r(v)| \leqq \Gamma N; \qquad \left| F'_\beta\left(\frac{a'}{q'}\right) \right| \leqq \sum_{v'} |r'(v')| \leqq \Gamma' N,$$

$$\left| \Phi_\beta\left(\frac{a'}{q'}\right) \right| \leqq \gamma_1\, q^l \sum_v |\varrho(v)| \leqq \gamma_1\, \Gamma N n^{l\sigma}$$

et

$$\left| \Phi'_\beta\left(\frac{a'}{q'}\right) \right| \leqq \gamma_1\, q^l \sum_{v'} |\varrho'(v')| \leqq \gamma_1\, \Gamma' N n^{l\sigma},$$

d'où il suit

$$\begin{aligned} \left| G_\beta\left(\frac{a'}{q'}\right) \right| &\leqq \left| F_\beta\left(\frac{a'}{q'}\right) F'_\beta\left(\frac{a'}{q'}\right) \right| + \left| \Phi_\beta\left(\frac{a'}{q'}\right) \Phi'_\beta\left(\frac{a'}{q'}\right) \right| \\ &\leqq c_{13}\, \Gamma \Gamma'\, N^2\, n^{2l\sigma}, \end{aligned}$$

par conséquent

$$\left| G_\alpha\left(\frac{a}{q}\right) G_\beta\left(\frac{a'}{q'}\right) \right| \leqq c_{14}\, \Gamma^2\, \Gamma'^2\, N^4\, n^{-m-1-4\sigma-\tau}.$$

De cette manière on trouve que la contribution U_6 est tout au plus égale à

$$c_{14}\, \Gamma^2\, \Gamma'^2\, N^4\, n^{-m-1-4\sigma-\tau} \sum_{\frac{a}{q}} \sum_{\frac{a'}{q'}} \int\limits_{\frac{a}{q}-N^{-1}n^{\tau}}^{\frac{a}{q}+N^{-1}n^{\tau}} \int\limits_{\frac{a'}{q'}-\frac{1}{2}}^{\frac{a'}{q'}+\frac{1}{2}} M(\beta-\alpha)\, d\alpha\, d\beta$$

$$\leqq c_{14}\, \Gamma^2\, \Gamma'^2\, N^4\, n^{-m-1-4\sigma-\tau} \,.\, n^{4\sigma} \,.\, 2N^{-1}\, n^{\tau} \,.\, 5n$$

en vertu de (14). Ainsi on obtient

$$U_6 \leqq c_{15}\, \Gamma^2\, \Gamma'^2\, N^3\, n^{-m}$$

et le théorème 1 est démontré.

Dans les applications la somme

$$\sum_{q=1}^{[n^{\tau}]} H(q, t)$$

nous fournirait beaucoup de difficultés. Afin d'éviter cet inconvénient, je déduirai sous certaines conditions supplémentaires une autre proposition dans laquelle cette somme est remplacée par un produit indépendant de N dont l'étude est plus facile. Pour la démonstration de cette nouvelle proposition j'ai besoin de deux propositions auxiliaires.

Lemme 4: *Soit* $Q > 0$; $x \geqq 2$; *désignons par* $A_x(Q)$ *le nombre des nombres naturels* $\leqq Q$ *dont chaque facteur premier est* $\leqq x$.

Pour tout nombre $\lambda \geqq \frac{1}{2}$ *on a*

$$A_x(Q) \leqq c_{16}\, Q^{\lambda} \prod_{p \leqq x} (1 - p^{-\lambda})^{-1},$$

où c_{16} *est une constante absolue.*

Démonstration: Je puis admettre que Q soit $\geqq 1$; sinon $A_x(Q)$ s'annule. Désignons par r le nombre des nombres premiers $\leqq x$.

Dans le cas particulier où $r = 1$, $A_x(Q)$ est le nombre des nombres $2^h \leqq Q$, où h est entier $\geqq 0$, par conséquent

$$A_x(Q) \leqq 1 + \frac{\log Q}{\log 2} \leqq c_{16} \sqrt{Q} \leqq \frac{c_{16}\, Q^{\lambda}}{1 - 2^{-\lambda}},$$

c_{16} désignant une constante absolue convenablement choisie.

Supposons maintenant que r soit > 1 et que le lemme soit déja démontré si l'on remplace r par $r-1$. Si g est le dernier, k l'avant-dernier nombre premier $\leqq x$, on a

$$A_x(Q) = A_g(Q) = \sum_{h=0}^{\infty} A_k\left(\frac{Q}{g^h}\right);$$

en effet, $A_k\left(\frac{Q}{g^h}\right)$ est le nombre des nombres naturels $\leqq Q$, dont chaque facteur premier est $\leqq g$, tandis que le fracteur g figure précisément h fois. D'après l'hypothèse d'induction on a

$$A_k\left(\frac{Q}{g^h}\right) \leqq c_{16}\,\frac{Q^\lambda}{g^{h\lambda}} \prod_{p \leqq k} (1-p^{-\lambda})^{-1},$$

d'où il suit

$$A_x(Q) \leqq c_{16}\, Q^\lambda \left\{ \prod_{p\leqq k} (1-p^{-\lambda})^{-1} \right\} \sum_{h=0}^{\infty} g^{-h\lambda}$$

$$= c_{16}\, Q^\lambda \left\{ \prod_{p\leqq k} (1-p^{-\lambda})^{-1} \right\}.\ (1-g^{-\lambda})^{-1}$$

$$= c_{16}\, Q^\lambda \prod_{p\leqq x} (1-p^{\;\lambda})^{-1}.$$

Lemme 5: *Le nombre $A_x(Q)$, déterminé dans le lemme précédent, est pour tout $Q>0$, pour tout $x \geqq 2$ et pour tout nombre positif $\varrho \leqq \frac{1}{2}$ inférieur à $c_{17}\, Q^{1-\varrho}\, e^{\frac{4x^\varrho}{\varrho}}$, où c_{17} est une constante absolue.*

Démonstration: Si l'on prend $\lambda = 1-\varrho$ (donc $\lambda \geqq \frac{1}{2}$), on obtient d'après le lemme précédent

$$A_x(Q) < c_{17}\, Q^{1-\varrho} \prod_{p\leqq x} (1-p^{\varrho-1})^{-1}.$$

On a pour tout nombre premier p

$$-\log(1-p^{\varrho-1}) \leqq p^{\varrho-1} + p^{2(\varrho-1)} + p^{3(\varrho-1)} + \dots$$

$$\leqq p^{\varrho-1} \sum_{h=0}^{\infty} 2^{-\frac{1}{2}h} = \frac{p^{\varrho-1}}{1-\frac{1}{2}\sqrt{2}} < 4p^{\varrho-1},$$

par conséquent

$$-\sum_{p\leqq x} \log(1-p^{\varrho-1}) < 4 \sum_{p\leqq x} p^{\varrho-1} \leqq 4 \sum_{s=2}^{[x]} s^{\varrho-1}$$

$$< 4\int_0^x u^{\varrho-1}\,du = \frac{4x^\varrho}{\varrho},$$

d'où le lemme 5 découle.

Dans la proposition 2 je supposerai que $H(q, t)$ possède la propriété multiplicative suivante: pour toute paire de nombres naturels q_1 et q_2 qui sont premiers entre eux et pour tout entier t on a

$$H(q_1, t)\ H(q_2, t) = H(q_1 q_2, t)\,. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

La fonction $H(q, t)$ définie par (9), possède certainement cette propriété multiplicative, si chaque couple de fractions irréductibles $\frac{a_1}{q_1}$ et $\frac{a}{q_2}$ qui sont $\geqq 0$ et < 1 et dont les dénominateurs q_1 et q_2 sont premiers entre eux, vérifie les relations

$$\left.\begin{aligned} \lambda\left(\frac{a_1}{q_1}\right)\lambda\left(\frac{a_2}{q_2}\right) &= \lambda\left(\frac{a_1}{q_1}+\frac{a_2}{q_2}\right) && \text{lorsque } \frac{a_1}{q_1}+\frac{a_2}{q_2}<1, \\ &= \lambda\left(\frac{a_1}{q_1}+\frac{a_2}{q_2}-1\right) && \text{lorsque } \frac{a_1}{q_1}+\frac{a_2}{q_2}\geqq 1, \end{aligned}\right\} . \qquad (19)$$

$$\left.\begin{aligned} \lambda'\left(\frac{a_1}{q_1}\right)\lambda'\left(\frac{a_2}{q_2}\right) &= \lambda'\left(\frac{a_1}{q_1}+\frac{a_2}{q_2}\right) && \text{lorsque } \frac{a_1}{q_1}+\frac{a_2}{q_2}<1, \\ &= \lambda'\left(\frac{a_1}{q_1}+\frac{a_2}{q_2}-1\right) && \text{lorsque } \frac{a_1}{q_1}+\frac{a_2}{q_2}\geqq 1. \end{aligned}\right\} . \qquad (20)$$

En effet, a_1 parcourant le système des entiers $\leqq 0$ et $< q_1$ qui sont premiers avec q_1 et a_2 parcourant le système des entiers $\leqq 0$ et $< q_2$ qui sont premiers avec q_2, le nombre a, définié par

$$\begin{aligned} \frac{a}{q_1 q_2} &= \frac{a_1}{q_1}+\frac{a_2}{q_2} && \text{lorsque } \frac{a_1}{q_1}+\frac{a_2}{q_2}<1, \\ &= \frac{a_1}{q_1}+\frac{a_2}{q_2}-1 && \text{lorsque } \frac{a_1}{q_1}+\frac{a_2}{q_2}\geqq 1, \end{aligned}$$

parcourt le système des entiers $\leqq 0$ et $< q_1 q_2$ qui sont premiers avec $q_1 q_2$, d'où il suit

$$\begin{aligned} H(q_1,t)H(q_2,t) &= \sum_{\substack{a_1=0 \\ (a_1,q_1)=1}}^{q_1-1} \sum_{\substack{a_2=0 \\ (a_2,q_2)=1}}^{q_2-1} \lambda\left(\frac{a_1}{q_1}\right)\lambda'\left(\frac{a_1}{q_1}\right)\lambda\left(\frac{a_2}{q_2}\right)\lambda'\left(\frac{a_2}{q_2}\right) e^{-2\pi i\left(\frac{a_1}{q_1}+\frac{a_2}{q_2}\right)t} \\ &= \sum_{\substack{a=0 \\ (a,q_1 q_2)=1}}^{q_1 q_2-1} \lambda\left(\frac{a}{q_1 q_2}\right)\lambda'\left(\frac{a}{q_1 q_2}\right) e^{-2\pi i\frac{at}{q_1 q_2}} \\ &= H(q_1 q_2, t). \end{aligned}$$

Proposition 2: *Supposons que les conditions de la proposition* 1 *soient vérifiées, que la fonction* $H(q, t)$ *possède la susdite propriété multiplicative* (18) *et qu'elle satisfasse pour tout nombre naturel* q, *tout entier* t *et tout nombre naturel* m *à l'inégalité*

$$|H(q,t)| \leqq \gamma_m q^{-1+\frac{1}{m}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

admettons en outre pour toute fraction irréductible

$$\left|\lambda\left(\frac{a}{q}\right)\lambda'\left(\frac{a}{q}\right)\right| \leqq \gamma_1 q^{-1-\varkappa}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

où $\varkappa$ *désigne un nombre positif indépendant de* $\frac{a}{q}$.

Sous ces conditions on a pour tout nombre positif m et pour tout nombre $\omega \geqq \frac{m}{2\varkappa}$

$$\sum_{t=2}^{[N]} \Big| L(t) - \Lambda(t) \prod_{p} \sum_{\zeta=0}^{\left[\frac{\omega \log\log t}{\log p}\right]} H(p^{\zeta}, t) \Big|^2 < c_{18}\, \Gamma^2\, \Gamma'^2\, N^3\, n^{-m},$$

où c_{18} *dépend uniquement de* m, l, $\varkappa$, ω *et des suites* (γ) *et* (η).

Démonstration: Prenons pour T l'intervalle fermé $(2, N)$ et appliquons le théorème 1, où m est remplacé par le plus petit entier $\geqq m$ et $\geqq \frac{m}{2\varkappa}$. Ainsi nous trouvons un nombre $\sigma \geqq \frac{m}{2\varkappa}$, dépendant uniquement de m, $\varkappa$ et de la suite (η), et aussi un nombre c_{19} dépendant uniquement de m, l, $\varkappa$ et des suites (γ) et (η) tels que

$$\sum_{t=2}^{[N]} \Big| L(t) - \Lambda(t) \sum_{q=1}^{[n^{\sigma}]} H(q, t) \Big|^2 < c_{19}\, \Gamma^2\, \Gamma'^2\, N^3\, n^{-m}.$$

Il suffit donc de déduire l'inégalité

$$\sum_{t=2}^{[N]} |\Lambda(t)|^2 \cdot \Big| \sum_{q=1}^{[n^{\sigma}]} H(q, t) - \prod_{p} \sum_{\zeta=0}^{\left[\frac{\omega \log\log t}{\log p}\right]} H(p^{\zeta}, t) \Big|^2 < c_{20}\, \Gamma^2\, \Gamma'^2\, N^3\, n^{-m};$$

dans cette démonstration $c_{20}, c_{21}, \dots, c_{45}$ désignent des nombres positifs, dépendant uniquement m, l, $\varkappa$, ω et des suites (γ) et (η).

La valeur absolue de $\varrho'(v')$ étant $\leqq \Gamma'$, on a

$$|\Lambda(t)| = \Big| \sum_{v+v'=t} \varrho(v)\, \varrho'(v') \Big| \leqq \Gamma' \sum_{v} |\varrho(v)| \leqq \Gamma\, \Gamma'\, N$$

en vertu de (10), de sorte qu'il suffit de démontrer

$$\sum_{t=2}^{[N]} \Big| \sum_{q=1}^{[n^{\sigma}]} H(q, t) - \prod_{p} \sum_{\zeta=0}^{\left[\frac{\omega \log\log t}{\log p}\right]} H(p^{\zeta}, t) \Big|^2 < c_{21}\, N\, n^{-m}. \quad . \quad (23)$$

Il suffit même de déduire l'inégalité

$$\sum_{\sqrt{N} < t \leqq N} \Big| \sum_{q=1}^{[n^{\sigma}]} H(q, t) - \prod_{p} \sum_{\zeta=0}^{\left[\frac{\omega \log\log t}{\log p}\right]} H(p^{\zeta}, t) \Big|^2 < c_{22}\, N\, n^{-m}. \quad . \quad . \quad (24)$$

En effet, si nous remplaçons dans (21) m par le plus petit entier $\geqq 2\omega$, nous obtenons

$$|H(q, t)| \leqq c_{23}\, q^{-1+\frac{1}{2\omega}},$$

$$\sum_{\zeta=1}^{\left[\frac{\omega \log\log t}{\log p}\right]} |H(p^{\zeta}, t)| \leqq c_{23} \sum_{\zeta=1}^{\infty} p^{\left(-1+\frac{1}{2\omega}\right)\zeta} \leqq c_{24}\, p^{-1+\frac{1}{2\omega}}.$$

Pour chaque nombre premier p tel que $c_{24}\, p^{-1+\frac{1}{2\omega}} \leqq \frac{1}{2}$ on a donc

$$\log\Big(1 + \sum_{\zeta=1}^{\left[\frac{\omega \log\log t}{\log p}\right]} H(p^{\zeta}, t)\Big) \leqq c_{25}\, p^{-1+\frac{1}{2\omega}},$$

par conséquent

$$\prod_{p} \sum_{\zeta=0}^{\left[\frac{\omega \log\log t}{\log p}\right]} H(p^{\zeta}, t) \leqq c_{26}\, e^{c_{25} \sum\limits_{p \leqq (\log t)^{\omega}} p^{-1+\frac{1}{2\omega}}}$$

$$\leqq c_{26}\, e^{c_{27}\sqrt{\log t}} \leqq c_{26}\, e^{c_{28}\sqrt{n}} \leqq c_{29}\, e^{\frac{1}{6} n} = c_{29}\, N^{\frac{1}{6}};$$

en outre on a

$$\Big|\sum_{q=1}^{[n^{\sigma}]} H(q, t)\Big| \leqq c_{23} \sum_{q=1}^{[n^{\sigma}]} q^{-1+\frac{1}{2\omega}} < c_{30}\, n^{\frac{\sigma}{2\omega}} < c_{31}\, N^{\frac{1}{6}},$$

d'où il suit

$$\sum_{2 \leqq t \leqq \sqrt{N}} \Big| \sum_{q=1}^{[n^{\sigma}]} H(q, t) - \prod_{p} \sum_{\zeta=0}^{\left[\frac{\omega \log\log t}{\log p}\right]} H(p^{\zeta}, t) \Big|^2 < c_{32} \sum_{2 \leqq t \leqq \sqrt{N}} N^{\frac{1}{3}}$$

$$< c_{33}\, N^{\frac{5}{6}} < c_{34}\, N n^{-m},$$

de sorte que (23) est une conséquence immédiate de (24).

Démontrons maintenant (24). La propriété multiplicative de $H(q, t)$ nous apprend

$$\sum_{q=1}^{[n^{\sigma}]} H(q, t) - \prod_{p} \sum_{\zeta=0}^{\left[\frac{\omega \log\log t}{\log p}\right]} H(p^{\zeta}, t) = \sum_{q}{}_{1} H(q, t) - \sum_{q}{}_{2} H(q, t) - \sum_{q}{}_{3} H(q, t);$$

$\sum\limits_{q}{}_{1}$ est étendu à tous les nombres naturels $q \leqq n^{\sigma}$ possédant au moins un diviseur $p^{\zeta} > (\log t)^{\omega}$; $\sum\limits_{q}{}_{2}$ est étendu à tous les entiers $q > n^{\sigma}$ et $\leqq N^{\frac{2}{9}}$ dont tous les diviseurs de la forme p^{ζ} sont $\leqq (\log t)^{\omega}$, tandis que $\sum\limits_{q}{}_{3}$ est étendu à tous les entiers $q > N^{\frac{2}{9}}$ dont tous les facteurs de la forme p^{ζ} sont $\leqq (\log t)^{\omega}$. Il suffit donc de démontrer

$$\sum_{\sqrt{N} < t \leqq N} \Big| \sum_{q}{}_{\tau} H(q, t) \Big|^2 \leqq c_{35}\, N n^{-m} \quad (\tau = 1,\ 2 \text{ et } 3); \quad . \quad . \quad (25)$$

en effet, le premier membre de (24) est alors $\leqq 9\, c_{35}\, N n^{-m}$.

On a

$$\left.\begin{aligned}\sum_{\sqrt{N}<t\leqq N}\Big|\sum_{q}{}_1 H(q,t)\Big|^2 &= \sum_{\sqrt{N}<t\leqq N}\sum_{q}{}_1\sum_{q'}{}_1 H(q,t)\,\overline{H}(q',t)\\ &= \sum_{q=1}^{[n^\sigma]}\sum_{q'=1}^{[n^\sigma]}\sum_{t}{}_4\, H(q,t)\,\overline{H}(q',t),\end{aligned}\right\}\qquad(26)$$

où w désigne la valeur complexe conjuguée de w et où $\sum_t{}_4$ est étendu à tous les entiers t de l'intervalle $\sqrt{N}<t\leqq N$ tels que chacun des deux nombres q et q' contienne au moins un diviseur $p^{\varkappa}>(\log t)^{\omega}$. La somme $\sum_t{}_4$ est donc étendue à tous les entiers d'un intervalle dépendant de q et q'. Si au moins un des deux nombres q et q' est $\leqq(\frac{1}{2}n)^{\omega}$, ce nombre est $\leqq(\log t)^{\omega}$ et la somme $\sum_t{}_4$ s'annule. Ainsi on obtient pour le premier membre de (26) la valeur

$$\sum_{\substack{(\frac{1}{2}n)^{\omega}<q\leqq n^{\sigma}\\(\frac{1}{2}n)^{\omega}<q'\leqq n^{\sigma}}}\ \sum_{t}{}_4\ \sum_{\substack{a=0\\(a,q)=1}}^{q-1}\ \sum_{\substack{a'=0\\(a',q')=1}}^{q'-1}\lambda^*\left(\frac{a}{q},\frac{a'}{q'}\right)e^{2\pi i\left(\frac{a'}{q'}-\frac{a}{q}\right)t}=U_1+U_2;$$

j'ai posé

$$\lambda^*\left(\frac{a}{q},\frac{a'}{q'}\right)=\lambda\left(\frac{a}{q}\right)\lambda'\left(\frac{a}{q}\right)\lambda\left(\frac{a'}{q'}\right)\lambda'\left(\frac{a'}{q'}\right);$$

U_1 est la contribution des termes tels que $q=q'$ et $a=a'$, tandis que U_2 est la contribution des autres termes. Puisque $\sum_t{}_4$ renferme moins de N termes, on a en vertu de (22)

$$|U_1|\leqq N\sum_{q>(\frac{1}{2}n)^{\omega}}\ \sum_{\substack{a=0\\(a,q)=1}}^{q-1}\left|\lambda^*\left(\frac{a}{q},\frac{a}{q}\right)\right.$$

$$\leqq N\gamma_1{}^2\sum_{q>(\frac{1}{2}n)^{\omega}}\ \sum_{a=0}^{q-1}q^{-2-2\varkappa}<c_{36}\,N\,n^{-2\varkappa\omega}\leqq c_{36}\,N\,n^{-m}.$$

Pour les termes figurant dans U_2 on a

$$0<\left|\frac{a'}{q'}-\frac{a}{q}\right|<1,\ \text{donc}\ \frac{1}{q\,q'}\leqq\left|\frac{a'}{q'}-\frac{a}{q}\right|\leqq 1-\frac{1}{q\,q'},$$

$$\left|\sum_{t}{}_4\, e^{2\pi i\left(\frac{a'}{q'}-\frac{a}{q}\right)t}\right|\leqq\frac{2}{\left|\sin\dfrac{\pi}{q\,q'}\right|}\leqq q\,q',$$

d'où il suit en vertu de (22)

$$|U_2| \leqq \sum_{q=1}^{[n^\sigma]} \sum_{q'=1}^{[n^\sigma]} \sum_{\substack{a=0\\(a,q)=1}}^{q-1} \sum_{\substack{a'=0\\(a',q')=1}}^{q'-1} q\,q' \left| \lambda^{\star}\left(\frac{a}{q}, \frac{a'}{q'}\right) \right|$$

$$= \left\{ \sum_{q=1}^{[n^\sigma]} \sum_{\substack{a=0\\(a,q)=1}}^{q-1} q \left| \lambda\left(\frac{a}{q}\right) \lambda'\left(\frac{a}{q}\right) \right| \right\}^2$$

$$\leqq \gamma_1{}^2 \left\{ \sum_{q=1}^{[n^\sigma]} q \right\}^2 \leqq \gamma_1{}^2\, n^{4\sigma} < c_{37}\, N\, n^{-m}.$$

Ainsi nous obtenons (25) pour $\tau = 1$.

En outre on a

$$\sum_{\sqrt{N} < t \leqq N} \Big| \sum_{q}{}_2\, H(q, t) \Big|^2 = \sum_{\substack{n^\sigma < q \leqq N^{\frac{2}{9}}\\ n^\sigma < q' \leqq N^{\frac{2}{9}}}} \sum_{t}{}_5\, H(q, t)\, \overline{H}(q', t);$$

$\sum_{t}{}_5$ est étendu à tous les entiers $t > \sqrt{N}$ et $\leqq N$ tels que tout nombre de la forme p^ξ qui est un diviseur d'au moins un des deux nombres q et q' soit $\leqq (\log t)^\omega$. Cette somme est donc étendue à tous les entiers d'un intervalle dépendant de q et q'. Le raisonnement précédent nous apprend que la contribution des termes tels que $q = q'$ et $a = a'$ est

$$\leqq N \gamma_1{}^2 \sum_{q > n^\sigma} q^{-1-2\varkappa} \leqq c_{38}\, N\, n^{-2\varkappa\sigma} \leqq c_{38}\, N\, n^{-m}$$

et que la contribution des autres termes est

$$\leqq \gamma_1{}^2 \left\{ \sum_{q=1}^{[N^{\frac{2}{9}}]} q \right\}^2 \leqq \gamma_1{}^2\, N^{\frac{8}{9}} < c_{39}\, N\, n^{-m}.$$

Ainsi nous obtenons (25) pour $\tau = 2$.

Considérons enfin (25) pour $\tau = 3$. Soit $Q \geqq 2\, N^{\frac{2}{9}}$. D'après le lemme précédent, appliqué avec $\varrho = \frac{1}{2\omega'}$ et $x = (\log t)^{\omega'}$, où $\omega' = \omega + 1$, la somme $\sum_{q}{}_3$ renferme moins de

$$c_{40}\, Q^{1 - \frac{1}{2\omega'}}\, e^{8\omega' \sqrt{\log t}} < c_{41}\, Q^{1 - \frac{1}{3\omega'}}$$

termes tels que $\frac{1}{2} Q < q \leqq Q$. La contribution de ces termes à $\sum_{q}{}_3 H(q, t)$ est en vertu de (21), où nous remplaçons m par le plus petit entier $\geqq 6\,\omega'$, inférieur à $c_{42}\, Q^{-\frac{1}{6\omega'}}$, d'où il suit

$$\Big| \sum_{q}{}_3\, H(q, t) \Big| < c_{43}\, N^{-\frac{1}{27\omega'}} < c_{44}\, n^{-\frac{1}{3} m},$$

par conséquent

$$\sum_{\sqrt{N} < t \leqq N} | \sum_{q}{}_{3} H(q, t) |^2 < c_{45} N n^{-m}.$$

Ainsi nous avons obtenu (25) pour $\tau = 3$, de sorte que la proposition est demontrée.

Je donnerai plusieurs applications de ce théorème.

Première application de la proposition 2.

Je me demande quels nombres naturels t peuvent être écrits sous la forme $p + \psi(x)$, où p est un nombre premier, x un nombre naturel,

$$\psi(x) = b x^g + \ldots \qquad (b > 0)$$

un polynôme donné du degré $g \geqq 1$ à coefficients entiers. Désignons par G le plus grand commun diviseur de tous les nombres $\psi(x) - \psi(0)$, où x est entier. Si le plus grand commun diviseur de $t - \psi(0)$ et G est supérieur à 1, la décomposition $t = p + \psi(x)$ impliquerait que le nombre premier

$$p = t - \psi(0) - (\psi(x) - \psi(0))$$

est divisible par, donc égal à ce plus grand commun diviseur. Ainsi nous obtenons: si le plus grand commun diviseur de $t - \psi(0)$ et G est un nombre > 1 qui n'est pas premier, t ne possède pas la forme $p + \psi(x)$. Si le plus grand commun diviseur de $t - \psi(0)$ et G est un nombre premier p', t peut posséder la forme $p + \psi(x)$, mais la seule possibilité est alors $p = p'$. Plus importante est la remarque suivante:

Proposition 3: *Presque tout nombre naturel t tel que $t - \psi(0)$ soit premier avec G, possède la susdite forme. A tout nombre naturel m correspond même un nombre c_{46}, dépendant uniquement de m et du choix du polynôme $\psi(x)$, tel que le nombre des exceptions $\leqq N$ soit pour tout $N \geqq 3$ inférieur à $c_{46} N (\log N)^{-m}$.*

Il va sans dire qu'il y a une infinité d'exceptions; tout nombre $\psi(a)$ tel qu'aucun des deux nombres $\psi(a) - \psi(a-1)$ et $\psi(a) - \psi(a+1)$ soit premier, est une exception; en effet $\psi(a) = p + \psi(x)$ entraînerait

$$p = (a - x) \cdot \frac{\psi(a) - \psi(x)}{a - x},$$

par suite $x = a \mp 1$.

Désignons par $F(t)$ le nombre des manières dont on peut mettre un entier donné $t \geqq 4$ sous la forme $t = p + \psi(x)$, et posons

$$\Phi(t) = g^{-1} b^{-\frac{1}{g}} \sum_{h=2}^{t-2} \frac{h^{-1+\frac{1}{g}}}{\log(t-h)} \quad \ldots\ldots\ldots \quad (27)$$

et

$$\Omega_\nu(t) = \prod_{p \leqq (\log t)^\nu} (1 + H(p, t)), \quad \ldots\ldots\ldots \quad (28)$$

où

$$H(p,t)=\frac{1}{p-1}\left\{1-\sum_{\substack{h=1\\ p/\psi(h)-t}}^{p}1\right\}. \qquad (29)$$

L'ordre de grandeur de $\Phi(t)$ dépend de celle de t, tandis que l'ordre de grandeur de $\Omega_\nu(t)$ dépend du caractère arithmétique de t.

$F(t)$ possède, comme nous verrons, pour presque tout entier $t \geqq 4$ la valeur approximative $\Phi(t)\,\Omega_\nu(t)$, où ν est assez grand. Cette remarque est évidente pour les entiers $t \geqq 4$ tels que le plus grand commun diviseur de $t-\psi(0)$ et G soit supérieur à 1. En effet, nous venons de voir que $F(t)$ est pour un tel entier t tout au plus égal à g, tandis qu'il y a au moins un nombre premier p, qui est un diviseur de $\psi(h)-\psi(0)$ $(h=1,2,\ldots,p)$ et de $t-\psi(0)$, donc de $\psi(h)-t$, d'où il suit

$$1+H(p,t)=1+\frac{1}{p-1}(1-p)=0,$$

par conséquent $\Phi(t)\,\Omega_\nu(t)=0$, si l'on prend $(\log t)^\nu \geqq p$.

La remarque, qui n'est point du tout manifeste pour les autres entiers $t \geqq 4$, résulte de la proposition suivante:

Proposition 4: *Si m est un nombre positif et si l'on a*

$$\nu > 2^{g-2}(3mg+2m+2g), \qquad (30)$$

presque tout entier $t \geqq 4$ tel que $t-\psi(0)$ soit premier avec G satisfait à la relation

$$F(t)=\left(1+\frac{\theta}{(\log t)^m}\right)\Phi(t)\,\Omega_\nu(t) \text{ où } |\theta|<1. \qquad (31)$$

Le nombre des exceptions $\leqq N$ est même pour tout $N \geqq 3$ inférieur à $c_{47}N(\log N)^{-m}$, où c_{47} dépend uniquement de m, ν et du choix du polynôme $\psi(x)$.

Il serait facile de remplacer dans cette proposition le second membre de (30) par un plus petit nombre dépendant uniquement de g et m, mais cela ne vaut pas la peine.

Observons que nous avons $\Omega_\nu(t)>0$ pour tout entier t tel que $t-\psi(0)$ soit premier avec G. En effet, sinon il y aurait au moins un nombre premier p avec

$$H(p,t)=-1, \quad \text{donc} \sum_{\substack{h=1\\ p/\psi(h)-t}}^{p}1=p,$$

de sorte que p serait un diviseur de $\psi(h)-t$ $(h=1,2,\ldots,p)$, par conséquent de tous les nombres $\psi(x)-t$, où x est un entier quelconque; p serait alors un diviseur de $t-\psi(0)$ et de G.

Il va maintenant sans dire que la proposition 3 est une conséquence immédiate du théorème précédent.

Comme je vais le démontrer dans le lemme suivant, la fonction $\Phi(t)$ possède pour tout entier $t \geqq 4$ la valeur approximative $\frac{b^{-\frac{1}{g}} t^{\frac{1}{g}}}{\log t}$ et l'écart est inférieur à $\frac{c_{48} b^{-\frac{1}{g}} t^{\frac{1}{g}}}{(\log t)^2}$, où c_{48} dépend uniquement de g.

Considérons le cas particulier où $\psi(x) = x^2$; en posant $\psi(x) = x^2$, $g = 2$, $m = \frac{1}{9}$ et $\nu = 5$ nous obtenons le résultat que le nombre des manières d'écrire t comme la somme d'un nombre premier et du carré d'un nombre naturel possède pour presque tout entier $t \geqq 4$ la valeur approximative

$$\frac{\sqrt{t}}{\log t} \prod_{2 < p \leqq (\log t)^5} \left(1 - \frac{\left(\frac{t}{p}\right)}{p-1}\right),$$

où $\left(\frac{t}{p}\right)$ désigne le symbole de LEGENDRE, c'est-à-dire

$$\begin{aligned} \left(\frac{t}{p}\right) &= 0 \quad \text{lorsque } t \text{ est divisible par } p; \\ &= 1 \quad \text{lorsque } t \text{ est un reste quadratique de } p; \\ &= -1 \quad \text{lorsque } t \text{ est un non-reste de } p. \end{aligned}$$

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§ 17. *Komitanten mit $\hat{B}$ und $\grave{C}$.*

Eine Syzygie dritter Art entsteht wieder aus einer projektiven Komitante K des Tensors a_{ijk}, die linear ist in

$$\hat{B}_{uv,w} = (\beta u')(\beta v')(Bw') \text{ und } \hat{C}^{xy,z} = (\gamma' x)(\gamma' y)(C' z) \, . \quad . \quad (152)$$

Wir können uns vorerst auf $\hat{B}$ allein beschränken. Dann verfügen wir für K über die Reihen (Vgl. (85))

$$\beta, B, a, b, \ldots, x, y, \ldots; a', b', \ldots, u', v', \ldots$$

Auch hier gilt das im § 9 Gesagte, demzufolge man von Klammerfaktoren absehen kann. Bei den Linearfaktoren reduzieren wir $(a\,b')$ wieder auf $(a\,b')^2$ und erhalten so die Reihen $A\;A'$, die wir jetzt (um Verwechslungen mit der Reihe B von (152) und mit D, E, F vorzubeugen) mit $MM' = NN' = \ldots$ bezeichnen. Dann ist nach (89) (MN') ein Reduzent und für K bleiben die Faktoren

$$\left. \begin{array}{c} (\beta a'), (\beta M'), (\beta u'); (Ba'), (BM'), (Bu'); (Ma'), (Mu'); (aM'), (au'); \\ (xM'), (xa'), (xu'). \end{array} \right\} \quad (153)$$

Aus den Faktoren ohne β, B erhalten wir wieder neben J und $(u' x)$ die Komitanten (93). Wir haben also nur noch die Komitanten K mit Reihen β, B zu ermitteln. Hier stellen wir in erster Linie fest: $(\beta M')$ und (BM') sind Reduzenten. Sei nämlich z. B.

$$K = (\beta M')(\beta u')(Bv')(Mw') = \hat{B}_{\varrho u, v}\, M^{\varrho}_{w} ,$$

so folgt aus (143)

$$K \equiv 0 \;\{ modd\ Syzygien\ F \}$$

und genau so für den Faktor (BM') nach (142).

Bei einem Faktor (Ba') haben wir

$$K = \hat{B}_{uv,\varrho}\, a^{\varrho xy}$$

und dies gibt nach (115) Reduktion auf Komitanten der Gestalt

$$\hat{B}_u^x = B_{u\varrho,\sigma}\, a^{\varrho\sigma x} = (\beta u')\,(\beta a')\,(Ba')\,(a'x),$$

also auf $(\beta a')\,(B a')$.

Für K bleiben also die Ansätze

$$\left.\begin{aligned} K_1 &= (\beta\xi')\,(\beta\eta')\,(Bw') \\ K_2 &= (\beta\xi')\,(\beta a')\,(Ba') \end{aligned} \quad \text{mit } \xi', \eta' = a', b', \ldots, u', v'. \right\} \quad . \quad . \quad (154)$$

In K_2 kann ξ' nur $= u'$ sein. Für $\xi' = b'$ hätten wir nämlich

$$K_2 = \hat{B}_b^x\, b^{yz} = \hat{B}_\varrho^x\, b^{\varrho yz},$$

was nach (112) auf $\grave{C}_b^x\; b^{yz} = \tilde{C}^{x,yz}$ reduzierbar ist (Vgl. (148)). Und dieses letztere ist wieder nach der zu (141) dualen Gleichung *modd F* zu reduzieren.

In K_1 können wir nicht $\xi' = a'$ und $\eta' = b'$ haben; wegen der zyklischen Symmetrie der $\grave{B}$ ist nämlich

$$K_1 = -(\beta\eta')\,(\beta w')\,(B\xi') - (\beta w')\,(\beta\xi')\,(B\eta'),$$

wodurch man auf den schon behandelten Fall mit $(B a')$ zurückkommt.

Somit bleiben in (154) nur die Ansätze:

$$\left.\begin{aligned} K_1' &= (\beta u')\,(\beta v')\,(Bw') = \hat{B}_{uv,w} \\ K_1'' &= (\beta u')\,(\beta a')\,(Bw') = B_{u\varrho,w}\, a^{\varrho\xi\eta} \\ K_2 &= (\beta u')\,(\beta a')\,(Ba')\,(a'\xi) = \hat{B}_{u\varrho,\sigma}\, a^{\varrho\sigma\xi} = \hat{B}_u^\xi \end{aligned} \right\} \quad . \quad . \quad . \quad (155)$$

wobei $\xi,\ \eta = M, \ldots, x,\ y, \ldots$

Hier können wir schliesslich noch die Fälle $\xi,\ \eta = M$ ausschliessen. Wäre nämlich z. B. in $K_2\ \xi = M$, also

$$K_2 = \hat{B}_{u\varrho,\sigma}\, a^{\varrho\sigma\tau}\, M_\tau^x,$$

so ist identisch in a_{ijk}

$$a^{\varrho\sigma\tau}\, M_\tau^x = -a^{\varrho x\tau}\, M_\tau^\sigma$$

und dann enthält also K_2 wieder den Faktor (BM'), was nach (142) Reduktion ergibt.

Zusammenfassend haben wir:

K ist eine Linearform von

$$\hat{B}_{uv,w};\ \hat{B}_{\varrho v,w}\, a^{\varrho xy};\ \hat{B}_u^x = \hat{B}_{u\varrho,\sigma}\, a^{\varrho\sigma x} \text{ und } \hat{C}^{xy,z} \quad . \quad . \quad . \quad (156)$$

mit Koeffizienten, die Polynome in J, $(u'\,x)$ *und Komitanten* (93) *sind.*

In (156) haben wir $\hat{C}^{\varrho y,z} a_{\varrho v w}$ und $\hat{C}^x_u$ weggelassen, da diese *modd F* auf die entsprechenden $\hat{B}$ reduziert werden können.

§ 18. *Die irreduziblen S_3.*

Ebenso wie in § 12 bei den Syzygien zweiter Art S_2 haben wir nach dem letzten Satze für die S_3 das Resultat: die linke Seite jeder S_3 kann *modd F* auf eine Linearform K der Ausdrücke (156) zurückgeführt werden. Die Koeffizienten dieser Linearformen K sind Polynome in J, $(u' x)$ und den Komitanten (93). Ueberdies können wir uns, was die Zahl der Reihen $x, y, \ldots, u', v', \ldots$ betrifft auf zwei Fälle beschränken:

α) K enthält nur u', v' und w'.

β) K enthält nur x, y und u, v.

Bei α) haben wir zunächst den Ansatz

$$K = \hat{B}_{uv,w} G_1 + \hat{B}_{vw,u} G_2 + \hat{B}_{wu,v} G_3 + \hat{B}_{uu,v} G_{12} + \hat{B}_{uu,w} G_{13} + \ldots = 0 \quad (157)$$

wobei die G_i und G_{ik} Polynome in J, $A^\varrho_u a_{\varrho v w}$ und a_{uvw} sind. Daraus folgt zunächst, da alle Glieder in K von gleichem Grade in u', v' und w' sein müssen, dass K entweder die Gestalt

$$K' = \hat{B}_{uv,w} G_1 + \hat{B}_{vw,u} G_2 + \hat{B}_{wu,v} G_3 \equiv 0 \,\{S^k_i\} \quad . \quad . \quad . \quad (158)$$

oder aber die Gestalt

$$K'' = \hat{B}_{uu,v} G_{12} + \hat{B}_{uu,w} G_{13} \equiv 0 \,\{S^k_i\} \quad . \quad . \quad . \quad . \quad (159)$$

haben muss.

In K' eliminieren wir $\hat{B}_{wu,v}$ nach (76) und haben (mit neuen G_1 und G_2):

$$K' = \hat{B}_{uv,w} G_1 + \hat{B}_{vw,u} G_2 = 0,$$

oder, ausgeschrieben in den S^k_i (Vgl. (74)):

$$K' = S^\varrho_u a_{\varrho v w} G_1 + S^\varrho_v (a_{\varrho u v} G_1 + a_{\varrho w u} G_2) + S^\varrho_w a_{\varrho v u} G_2 \equiv 0 \,\{S^k_i\} \quad . \quad (160)$$

Wir setzen nun, mit Zahlenkoeffizienten α_{pqr}:

$$G_1 = \Sigma\, \alpha_{pqr} J^p (A^\varrho_u a_{\varrho v w})^q (a_{uvw})^r = \Sigma\, \alpha_{pqr} (S^\mu_\lambda S^\lambda_\mu)^p \,.\, (S^\varrho_u\, a_{\varrho v w})^q \,.\, (a_{uvw})^r.$$

Da G_1 homogen von gleichem Grade in u, v, w und homogen in den a_{ijk} ist, haben wir

$$q + r = const. \quad \text{und} \quad 4p + 3q + r = const.$$

d. h. wir können, wenn m und n für alle Glieder von K' Konstanten bedeuten, setzen:

$$q = m - 2p, \quad r = n - m + 2p,$$

also

$$\begin{aligned} G_1 &= \Sigma\, \alpha_{pqr} J^p (A^\varrho_u\, a_{\varrho v w})^{m-2p} (a_{uvw})^{n-m+2p} = \\ &= \alpha_0 (S^\varrho_u\, a_{\varrho v w})^m (a_{uvw})^{n-m} + \alpha_1 J \,.\, (S^\varrho_u\, a_{\varrho v w})^{m-2} (a_{uvw})^{n-m+2} + \ldots . \end{aligned}$$

und ebenso G_2 mit Koeffizienten $\beta_0, \beta_1, \ldots$. Setzen wir dies in (160) ein, wählen $i \neq k$ und ordnen K' nach Potenzen von S^k_i, so ist $(S^k_i)^{m+1}$ die höchste auftretende Potenz; der dazugehörige Koeffizient wird

$$\begin{aligned} &(u^i a_{kvw} + v^i a_{kuw}) [\alpha_0 (u^i a_{kvw})^m (a_{uvw})^{n-m} + \alpha_1 (2\, S^i_k) (u^i a_{kvw})^{m-2} (a_{uvw})^{n-m+2} + \ldots] + \\ &+ (v^i a_{kuw} + w^i a_{kvu}) [\beta_0 (u^i a_{kvw})^m (a_{uvw})^{n-m} + \beta_1 (2\, S^i_k) (u^i a_{kvw})^{m-2} (a_{uvw})^{n-m+2} + \ldots] \end{aligned}$$

und dies muss für alle S^i_k verschwinden. Dies gibt für $j = 0, i, 2, \ldots$:

$$\alpha_j (u^i a_{kvw} + v^i a_{kuw}) + \beta_j (v^i a_{kuw} + w^i a_{kvu}) \equiv 0 \quad \{u, v, w\}.$$

Setzt man hier den Koeffizienten von u^r gleich Null, so entsteht für $r \neq i$:

$$\alpha_j\, v^i a_{krw} + \beta_j (v^i a_{krw} + w^i a_{kvr}) = 0.$$

Und hier gibt der Koeffizient von v^s für $s \neq i$:

$$\beta_j\, w^i a_{ksr} = 0$$

d. h. $\beta_j = 0$ und ebenso $\alpha_j = 0$. Es ist also in (160) $G_1 = G_2 = 0$, d. h. K' verschwindet bereits identisch in den $\hat{B}$.

Bei K'' von (159) haben wir, da wieder G_{12} und G_{13} homogen von gleichem Grade in u, v, w sind: $G_{12} = G_{13} = 0$, also auch

$$K'' = 0 \,\{\hat{B}\}$$

womit der Fall α) erledigt ist.

Bei β) ergibt sich nach (156) analog zu (157) der Ansatz:

$$\left.\begin{aligned} K = \hat{B}_{uu,v}\, G_{12} + \hat{B}_{vv,u}\, G_{21} + \\ + \hat{B}_{\varrho u,u}\, a^{\varrho x y}\, G_1 + \hat{B}_{\varrho v,v}\, a^{\varrho x y}\, G_2 + \hat{B}_{\varrho u,v}\, a^{\varrho x y}\, G_3 + \hat{B}_{\varrho v,u}\, a^{\varrho x y}\, G_4 + \\ + \hat{B}^x_u\, G_5 + \hat{B}^x_v\, G_6 + \hat{B}^y_u\, G_7 + \hat{B}^y_v\, G_8, \end{aligned}\right\} \tag{161}$$

wobei jetzt die Koeffizienten G Polynome sind in

$$J,\ (u' x),\ (v' x),\ (u' y),\ (v' y),\ S^x_u,\ S^x_v,\ S^y_u,\ S^y_v. \quad \ldots \tag{162}$$

Zählt man in (161) den Grad von K bezüglich u, v ab, so sieht man, dass er in den beiden ersten Termen um zwei Einheiten höher ist als in den übrigen. (161) zerfällt somit in

$$K_1 = \hat{B}_{uu,v}\, G_{12} + \hat{B}_{vv,u}\, G_{21} \equiv 0 \quad \{S_i^k\} \;. \quad . \quad . \quad . \quad . \qquad (163)$$

und in

$$K_2 = \hat{B}_{\varrho u,u}\, a^{\varrho x y}\, G_1 + \ldots + \hat{B}_v^y\, G_8 \equiv 0 \quad \{S_i^k\} \quad . \quad . \quad . \qquad (164)$$

Bei K_1 von (163) haben wir wegen $\hat{B}_{uu,v} = 2\, S_u^\varrho\, a_{\varrho u v}$:

$$\tfrac{1}{2}\, K_1 = S_u^\varrho\, a_{\varrho u v}\, G_{12} + S_v^\varrho\, a_{\varrho v u}\, G_{21} \equiv 0 \quad \{S_i^k\} \quad . \quad . \quad . \qquad (165)$$

Wählen wir hier $i \neq u, v$, so muss K_1 identisch in S_i^k verschwinden. Nach (165) und (162) tritt aber S_i^k dann nur im Argument J von G_{12} und G_{21} auf. Setzen wir also

$$G_{12} = M_0 + M_1\, J + M_2\, J^2 + \ldots, \qquad G_{21} = N_0 + N_1\, J + N_2\, J^2 + \ldots,$$

so folgen

$$K_{1j} = S_u^\varrho\, a_{\varrho u v}\, M_j + S_v^\varrho\, a_{\varrho v u}\, N_j \equiv 0 \quad \{S_i^k\} \quad (j = 0, 1, 2, \ldots) \quad . \qquad (166)$$

wobei jetzt M_j und N_j nur noch Polynome von

$$S_u^x,\; S_v^x,\; S_u^y,\; S_v^y,\; (u'x),\; (u'y),\; (v'x),\; (v'y)$$

sind.

In K_{1j} suchen wir den Koeffizienten von S_u^z mit $z \neq x, y, u, v$, sodass also S_u^z nicht in M_j oder N_j anwesend ist. Es wird dann

$$a_{zuv}\, M_j = 0, \text{ d. h. } M_j = 0 \text{ und ebenso auch } N_j = 0.$$

Der Ansatz (163) gibt daher

$$G_{12} = G_{21} = 0.$$

§ 19. *Reduktion von K_2.*

Nun bleibt noch (164) zu erledigen. In den S_i^k ausgeschrieben haben wir

$$\left.\begin{aligned} & S_u^\sigma\, a_{\sigma\varrho u}\, b^{\varrho x y}\, G_1 + S_v^\sigma\, a_{\sigma\varrho v}\, b^{\varrho x y}\, G_2 + (S_\varrho^\sigma\, a_{\sigma u v}\, b^{\varrho x y} + S_u^\sigma\, a_{\sigma\varrho v}\, b^{\varrho x y}) \,.\, G_3 + \ldots \\ & \quad + (S_u^\tau\, a_{\tau\varrho\sigma} + S_\varrho^\tau\, a_{\tau u \sigma})\, b^{x\varrho\sigma}\, G_5 + \ldots . \qquad \equiv 0 \,\{S_i^k\}. \end{aligned}\right\} (167)$$

Hier enthält der Term mit G_1 einen Faktor $(ab') = a_\varrho b^\varrho$, kann also nach (57*b*) reduziert werden. Es wird

$$S_u^\sigma\, a_{u\sigma\varrho}\, b^{\varrho x y} = \tfrac{1}{4}\,(S_\sigma^y\, S_u^\sigma\, (u'x) - S_\sigma^x\, S_u^\sigma\, (u'y)).$$

Analoge Reduktionen bei den weiteren Termen geben:

$$\left.\begin{aligned}
&\tfrac{1}{4}\,[S^y_\sigma\, S^\sigma_u\,(u'x)-S^x_\sigma\, S^\sigma_u\,(u'y)]\,.\,G_1+\tfrac{1}{4}\,[S^y_\sigma\, S^\sigma_v\,(v'x)-S^x_\sigma\, S^\sigma_v\,(v'y)]\,.\,G_2+\\
&+S^\sigma_\varrho\, a_{\sigma u v}\, b^{\varrho x y}\,(G_3-G_4)+\\
&+\tfrac{1}{4}\,[S^x_v\, S^y_u+S^y_\sigma\, S^\sigma_u\,(v'x)-S^x_\sigma\, S^\sigma_u\,(v'y)-S^y_v\, S^x_u]\,.\,G_3+\\
&+\tfrac{1}{4}\,[S^x_u\, S^y_v+S^y_\sigma\, S^\sigma_v\,(u'x)-S^x_\sigma\, S^\sigma_v\,(u'y)-S^y_u\, S^x_v]\,.\,G_4+\\
&+[\tfrac{3}{2}\,S^\sigma_u S^x_\sigma-\tfrac{1}{4}J(u'x)]\,.\,G_5+\ldots+[\tfrac{3}{2}\,S^\sigma_v S^y_\sigma-\tfrac{1}{4}J(v'y)]\,.\,G_8\equiv 0\ \{S^k_i\}
\end{aligned}\right\}\quad(168)$$

Vergleichen wir hier das dritte und vierte Glied, so sehen wir, dass beim dritten der Grad in den S^k_i um eine Einheit niedriger ist als beim vierten. Es folgt also $G_3=G_4$ und das dritte Glied fällt weg.

Wir entwickeln dann wieder nach Potenzen von J:

$$G_j=G_{j0}+J\,.\,G_{j1}+J^2\,.\,G_{j2}+\ldots\qquad(j=1,2,\ldots,8)$$

und haben statt (168):

$$\left.\begin{aligned}
&[S^y_\sigma\, S^\sigma_u\,(u'x)-S^x_\sigma\, S^\sigma_u\,(u'y)]\,.\,(G_{10}+JG_{11}+\ldots)+\ldots\\
&+[S^y_\sigma S^\sigma_v\,(u'x)+S^y_\sigma S^\sigma_u\,(v'x)-S^x_\sigma S^\sigma_v\,(u'y)-S^x_\sigma S^\sigma_u\,(v'y)]\,.\,(G_{30}+JG_{31}+\ldots)+\ldots\\
&+[6\,S^\sigma_u\, S^x_\sigma-J\,(u'x)]\,.\,(G_{50}+JG_{51}+\ldots)+\ldots\equiv 0\ \{S^k_i\}
\end{aligned}\right\}\quad(169)$$

oder kürzer:

$$AG_1+BG_2+CG_3+[D_5-J\,.\,(u'x)]\,.\,G_5+\ldots+[D_8-J\,.\,(v'y)]\,.\,G_8\equiv 0,\quad(169a)$$

wobei die Bedeutung von $A, B, C, D_5, \ldots, D_8$ aus (169) ersichtlich ist. Wählen wir hier i und k verschieden von x, y, u und v, so ist S^k_i nach (162) nur noch in J enthalten und (169) muss also identisch in J verschwinden. Dies gibt, wenn man die Koeffizienten von $J^0, J^1, J^2, \ldots$ gleich Null setzt:

$$\left.\begin{aligned}
H_0&=AG_{10}+BG_{20}+CG_{30}+D_5\,G_{50}+\ldots+D_8\,G_{80}=0\\
H_1&=AG_{11}+BG_{21}+CG_{31}+(D_5\,G_{51}-(u'x)\,G_{50})+\ldots+\\
&\qquad\qquad+(D_8\,G_{81}-(v'y)\,G_{80})=0\\
H_2&=AG_{12}+BG_{22}+CG_{32}+(D_5\,G_{52}-(u'x)\,G_{51})+\ldots+\\
&\qquad\qquad+(D_8\,G_{82}-(v'y)\,G_{81})=0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
H_j&=AG_{1j}+BG_{2j}+CG_{3j}+(D_5\,G_{5j}-(u'x)\,G_{5,j-1})+\ldots+\\
&\qquad\qquad+(D_8\,G_{8j}-(v'y)\,G_{8,j-1})=0.
\end{aligned}\right\}\quad(170)$$

Für $j=0$ haben wir, ausgeschrieben:

$$\left.\begin{aligned}
H_0&=[S^y_\sigma S^\sigma_u\,(u'x)-S^x_\sigma S^\sigma_u\,(u'y)]\,G_{10}+\ldots+[S^y_\sigma S^\sigma_v(u'x)+\ldots]\,G_{30}+\\
&+6\,S^\sigma_u\, S^x_\sigma\, G_{50}+6\,S^\sigma_v\, S^x_\sigma\, G_{60}+6\,S^\sigma_u\, S^y_\sigma\, G_{70}+6\,S^\sigma_v\, S^y_\sigma\, G_{80}=0.
\end{aligned}\right\}\quad(171)$$

Wählen wir hier $i \neq u, v$, so gibt der Koeffizient von S_i^y:

$$S_u^i (u'x)\, G_{10} + S_v^i (v'x)\, G_{20} + [S_v^i (u'x) + S_u^i (v'x)]\, G_{30} + 6\, S_u^i\, G_{70} + 6 S_v^i\, G_{80} = 0.$$

Ist überdies $i \neq x, y$, so kommen S_u^i und S_v^i nicht in den Koeffizienten G vor und es folgen

$$\left.\begin{aligned} -6\, G_{70} &= (u'x)\, G_{10} + (v'x)\, G_{30} \\ -6\, G_{80} &= (v'x)\, G_{20} + (u'x)\, G_{30} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (171a)$$

und auf ganz analoge Weise, indem wir wieder $i \neq u, v$ wählen und den Koeffizienten von S_i^x gleich Null setzen:

$$\left.\begin{aligned} +6\, G_{50} &= (u'y)\, G_{10} + (v'y)\, G_{30} \\ +6\, G_{60} &= (v'y)\, G_{20} + (u'y)\, G_{30} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (171b)$$

Setzt man dies in (171) ein, so wird $H_0 = 0$; setzt man es in H_1 ein, so entsteht

$$\left.\begin{aligned} H_1 = [\qquad\quad]\, G_{11} + [\qquad\quad]\, G_{21} + [\qquad\quad]\, G_{31} + \\ + 6\, S_u^\sigma\, S_\sigma^x\, G_{51} + 6\, S_v^\sigma\, S_\sigma^x\, G_{61} + 6\, S_u^\sigma S_\sigma^y\, G_{71} + 6\, S_v^\sigma S_\sigma^y\, G_{81} = 0. \end{aligned}\right\} (172)$$

Hieraus, auf dieselbe Weise, die zu den Gleichungen (171) führte:

$$\begin{aligned} -6\, G_{71} &= (u'\,x)\, G_{11} + (v'\,x)\, G_{31} & 6\, G_{51} &= (u'y)\, G_{11} + (v'\,y)\, G_{31} \\ -6\, G_{81} &= (v'\,x)\, G_{21} + (u'\,y)\, G_{31} & 6\, G_{61} &= (v'\,y)\, G_{21} + (u'\,y)\, G_{31} \end{aligned}$$

Mit diesen Werten wird (172) befriedigt und in H_2 eingesetzt ergeben sie gleichgebaute Ausdrücke. Allgemein wird dann

$$\left.\begin{aligned} 6\, G_{5j} &= (u'\,y)\, G_{1j} + (v'\,y)\, G_{3j} & -6\, G_{7j} &= (u'\,x)\, G_{1j} + (v'\,x)\, G_{3j} \\ 6\, G_{6j} &= (v'\,y)\, G_{2j} + (u'\,y)\, G_{3j} & -6\, G_{8j} &= (v'\,x)\, G_{2j} + (u'\,x)\, G_{3j} \end{aligned}\right\} \quad . \quad (173)$$

sodass wir schliesslich für die G_s selbst haben:

$$\left.\begin{aligned} G_5 &= \tfrac{1}{6} [(u'\,y)\, G_1 + (v'\,y)\, G_3] & G_7 &= -\tfrac{1}{6} [(u'\,x)\, G_1 + (v'\,x)\, G_3] \\ G_6 &= \tfrac{1}{6} [(v'\,y)\, G_2 + (u'\,y)\, G_3] & G_8 &= -\tfrac{1}{6} [(v'\,x)\, G_2 + (u'\,x)\, G_3] \end{aligned}\right\} \quad . \quad (174)$$

Setzen wir dies in (164) ein, so ergibt sich:

$$\left.\begin{aligned} K_2 = [\hat{B}_{\varrho u, u}\, \mathrm{a}^{\varrho x y} + \tfrac{1}{6} \hat{B}_u^x\, (u'\,y) - \tfrac{1}{6} \hat{B}_u^y\, (u'\,x)]\, .\, G_1 + [\qquad\qquad]\, .\, G_2 + \\ + [\hat{B}_{\varrho u, v}\, \mathrm{a}^{\varrho x y} + \hat{B}_{\varrho v, u}\, \mathrm{a}^{\varrho x y} + \tfrac{1}{6} \hat{B}_u^x (v'\,y) + \\ + \tfrac{1}{6} \hat{B}_v^x (u'\,y) - \tfrac{1}{6} \hat{B}_u^y (v'\,x) - \tfrac{1}{6} \hat{B}_v^y (u'\,x)]\, .\, G_3. \end{aligned}\right\} \quad . \quad (175)$$

Schreiben wir dies kürzer wie folgt:

$$K_2 = A_1\, G_1 + A_2\, G_2 + A_3\, G_3,$$

dann ist wegen

$$\hat{B}_{\varrho u, v} + \hat{B}_{\varrho v, u} = - \hat{B}_{u v, \varrho}$$

$$- 24 A_3 = 24 \grave{B}_{u v, \varrho} \, a^{\varrho x y} - 4 [\hat{B}^x_u (v' y) + \hat{B}^x_v (u' y) - \grave{B}^y_u (v' x) - \grave{B}^y_v (u' x)]$$

und also nach (115)

$$- 24 A_3 = D^{xy}_{uv} - \tfrac{1}{6} [D^x_u (v' y) + D^x_v (u' y) - D^y_u (v' x) - D^y_v (u' x)],$$

woraus schliesslich nach (125) und (126) folgt:

$$A_3 = F^{xy}_{uv} + F^{xy}_{vu} + \tfrac{1}{6} [F^x_u (v' y) + F^x_v (u' y) - F^y_u (v' x) - F^y_v (u' x)] \ . \quad (176)$$

Setzen wir hierin $u = v$, so ergeben sich

$$A_1 = F^{xy}_{uu} + \tfrac{1}{6} [F^x_u (u' y) - F^y_u (u' x)], \quad A_2 = F^{xy}_{vv} + \tfrac{1}{6} [F^x_v (v' y) - F^y_v (v' x)] \quad (178)$$

womit auch

$$K_2 \equiv 0 \ (modd \ F)$$

bewiesen ist.

Chemistry. — *Flow potentials, electro-endosmosis and electrophoresis with platinum.* By H. R. KRUYT and J. OOSTERMAN.

(Communicated at the meeting of March 26, 1938.)

In previous communications 1) we described how the measuring of the flow potential on a platinum capillary is complicated by occurring polarization phenomena but that ultimately it appeared that the potential in the flow cell became zero if the flowing liquid was composed of 10^{-4} n.HCl + + 1 $^0/_{00}$ quinhydrone. The question presents itself what the electrokinesis of platinum will be like on examination of electro-endosmosis or electrophoresis.

Electro-endosmosis with platinum plugs.

A platinum plug is to be preferred to a single capillary, since the quantity of liquid replaced electro-endosmotically in the first case is much larger than in the second, which highly benefits the accuracy of the measurements.

Fig. 1 reproduces the employed apparatus.

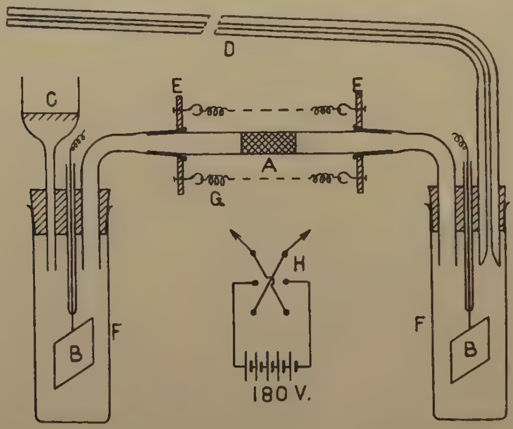

Fig. 1.

The plug *A* is pressed into a glass tubs of 0.6 cm diameter which via rectangularly bent small tubes with normal ground-in stoppers (held together

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1) Proc. Royal Acad. Amsterdam, **40**, 404 (1937). Doctoral thesis of J. OOSTERMAN (Utrecht 1937) and an elaborate article of the two of us to be published soon in Koll. Beihefte.

by means of the coil-springs *G* fastened to the ebonite flanges *E*) is connected with vessels *F* (flocculation glasses closed with carefully boiled rubber stoppers) in which are situated the strongly platinized platinum electrodes *B*. In our experiments these were connected via a commutator *H* with a cell of 180 V. Formation of gas on these electrodes was never observed. The funnel *C* serves, by adding drops of liquid, to bring the meniscus in the capillary *D* to a suitable point for reading. The shifting of the meniscus was observed by means of a telescope.

We tested the usefulness of the apparatus by first placing a plug of cotton wool in it; this was first boiled in distilled water. The electro-endosmotic water transport now was so large that within 8 sec. the meniscus in *D* shifted along 10 divisions of the scale on the telescope (27 divisions = 5 mm.). Since the meniscus in equilibrium in 100 sec. showed no greater shift than 0.4 division of the scale, an electro-endosmotic water transport may certainly be indicated which amounts to 1 % of that observed with the plug of cotton wool.

The preparation of the platinum plugs took place as follows: Platinum remnants were dissolved in nitrohydrochloric acid. The solution thus obtained was dried by evaporation, dissolved in hydrochloric acid and filtered, after which treatment the platinum was reduced in alkali with formaldehyde. The platinum then separates as a very fine black powder, which after filtration was successively boiled with dilute HCl, distilled water, dilute HNO_3, distilled water. Subsequently it was thoroughly rinsed with hot distilled water.

In the central glass tube of the apparatus a strong platinum wire was fastened by fusion, perpendicular to the length of the tube. On this was brought successively a small plug of platinum gauze, the finely divided platinum forming the actual plug, and another small plug of gauze to shut it off.

The platinum plug was made in various ways. Successively we prepared and examined:

a. a very strongly compressed plug of the extremely fine platinum black;

b. a very loosely compressed plug of the exceedingly fine platinum black;

c. a very strongly compressed plug of platinum sponge (for this purpose the platinum black was intimately mixed in an agate mortar with NH_4Cl and annealed in a tube of Pyrex glass till all NH_4Cl was sublimated. Thus we obtained a highly porous platinum mass which we boiled with HCl and distilled water);

d. a very loosely compressed plug of the same platinum sponge.

On application of a voltage up to 180 Volts the result of these experiments was perfectly negative. An eventual electro-endosmotic transport with platinum sponge consequently is certainly less than 1 % of that with cotton wool.

Electrophoresis with platinum sols.

Platinum sols were prepared according to Bredig's method of dispersion. The measurements were taken by the method of Kruyt and van Gils [1]).

The previous history of the measured sols was the following:

Two sols, I and II, were prepared by dispersion of *Pt*-electrodes in twice distilled water. To portions of these sols were added equal volumes of solutions, in such a way that Ia and II*a* contained 10^{-4} n.HCl as final concentration and I*b* and II*b* 10^{-4} n.HCl + 1 $^0/_{00}$ quinhydrone. In a similar way sol IIc was prepared, which contained no HCl but 1 $^0/_{00}$ quinhydrone.

Further, two sols, III and IV, were prepared by electrical dispersion in 10^{-4} n.HCl. Part of these sols again obtained a final concentration of 10^{-4} n.HCl + 1 $^0/_{00}$ quinhydrone, III*a* by dissolving the calculated amount of quinhydrone in the sol, IV*a* by diluting with an equal volume of a solution of 10^{-4} HCl + 2 $^0/_{00}$ quinhydrone [2]).

In the table the results have been given. The electrophoretic velocity E.V. is given in μ per volt. cm. sec.

No.	Dispersed in	Final concentration	E.V.	No.	Dispersed in	Final concentration	E.V.
I	water	0	2.75	II	water	0	2.62
Ia	,,	10^{-4} n. HCl	0.73	IIa	,,	10^{-4} n. HCl	1.07
I*b*	,,	10^{-4} n. HCl + $1^0/_{00}$ quinh.	2.52	II*b*	,,	10^{-4} n. HCl + $1^0/_{00}$ quinh.	2.94
				IIc	,,	$1^0/_{00}$ quinh.	3.04
III	10^{-4} n. HCl	10^{-4} n. HCl	2 12	IV	10^{-4} n. HCl	10^{-4} n. HCl	1.60
IIIa	,,	10^{-4} n. HCl + $1^0/_{00}$ quinh.	2 50	IV*a*	,,	10^{-4} n. HCl + $1^0/_{00}$ quinh.	2.54

The following should be observed here.

The sols I*a* and II*a* with their low E.V. are unstable; after 24 hours they are completely flocculated. If, however, to part of sol Ia the calculated amount of quinhydrone is added, so that a final concentration of 1 $^0/_{00}$ is attained, the E.V is raised. After 30 minutes it is 1.59, the next day 2.32, and the sol is perfectly stable.

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[1]) H. R. Kruyt and G. E. van Gils, Koll. Beih. **45**, 60 (1936).

[2]) The sols I*a* and I*b* consequently have half the sol concentration of sol I, similarly II*a*, II*b* and IIc half the sol concentration of sol II, and IV*a* that of IV. However, IIIa has the same sol concentration as III.

Discussion of the results.

The following questions have to be answered:

1. Why is the electrokinesis zero in measurements of electro-endosmosis on metal capillaries, and in measurements of flow potentials on metal capillaries, when polarization is excluded, while it becomes clearly manifest in case of glass capillaries?

2. Why does electrokinesis become manifest in case of electrophoresis of platinum sols, but not in measurements of flow potentials and electro-endosmosis on platinum capillaries?

3. Why has in case of electrophoresis of the sol the polarization no disturbing influence, while it has this effect on measurements of flow potentials?

The answers to these questions may be easily found if we try of form a picture of the lines of electric force in the system.

In fig. 2 the course is reproduced of the lines of electric force in case of a practically non-conductive capillary: they go through the tube and consequently the potential difference may set the opposite ions of the double layer into motion, which occasions electro-endosmosis. In fig. 3 the lines of electric force are given for a metal capillary, whose resistance is far smaller than that of the flowing liquid. Although the double layer is present, it cannot give rise to electro-endosmosis.

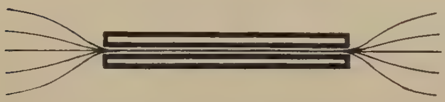

Fig. 2.

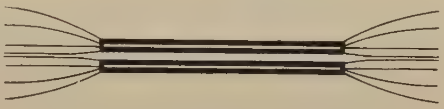

Fig. 3.

With the flow potential measurements the opposite ions are carried along by the hydrodynamic flow and a tension is formed between the two electrodes; but this E.M.F. causes a flow of the electricity in opposite direction. This, however, follows the reproduced lines of electric force: the conduction in the liquid takes place by means of ions, which must discharge at the ends of the capillary, causing a tension of poralization. If we prevent this from being formed by the presence of quinhydrone and hydrochloric acid, the flow potential according to the formula of VON SMOLUCHOWSKI [1]) appears to be immeasurably small.

The lines of electric force in the case of electrophoresis of highly conductive particles have been drawn in fig. 4, those for isolating particles in fig. 5. However different the course may be, in both cases the potential may attack the double layer and consequently electrophoresis will set in; at any rate if the atmosphere of the opposite ions is not entirely compressed in the immovable part of the double layer. This case was approximated with the sols I*a* and II*a* by the addition of 10^{-4} n.HCl to the formed sol.

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[1]) See the lit. cited under [1]) p. 370.

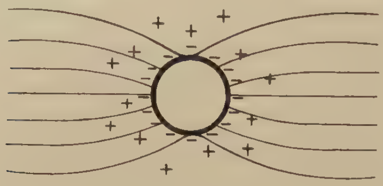

Fig. 4.

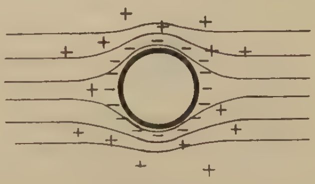

Fig. 5.

If, however, quinhydrone is added to this system, the ε-potential at the phase boundary platinumwater is radically modified and thus ζ is raised. That is why the sols I*b* and II*b* are much stabler than I*a* and II*a* respectively. Finally soll II*c* contains quinhydrone but no HCl and is in the most favourable condition with the greatest E.V.

That the sols dispersed in 10^{-4} n. HCl are stable, in contrast with the sols I*a* and II*a*, is due to the different constitution of the double layer; in all probability they contain chlor-platinum complexes as inner covering of the double layer.

Polarisation occurs no doubt at both sides of an electrophoretically migrating metal particle; but the demensions of the colloidal particle are too small that this phenomenon causes an appreciable effect on the migration. If it were possible to prepare a platinum sol with long needle shaped particles, such an effect of retardation would no doubt appear.

Utrecht, VAN 'T HOFF-*Laboratory of the*
State University.

Chemistry. — *Researches on Fat metabolism* **XI**. *Injection experiments on dogs with sodium salts of normal saturated dicarboxylic acids.* By P. E. VERKADE, J. VAN DER LEE, A. J. S. VAN ALPHEN and M. ELZAS.

(Communicated at the meeting of March 26, 1938.)

§ 1. It was shown in communication VIII of this series ¹) that sodium salts of normal saturated dicarboxylic acids supplied *per os* to one and the same dog under as nearly as possible the same conditions are excreted in the urine to an extent which decreases very typically on ascent of the homologous series. We have now repeated these experiments — the intention of doing so had already been announced — with the understanding that the sodium salts were supplied by injection. In the first mentioned work there is the possibility that the results were influenced by possible differences in the resorbability of the dicarboxylic acids or of their sodium salts by the intestinal wall; this is obviously excluded in the injection experiments described below.

§ 2. The healthy adult dogs Polly (about 7 kg, ♂) and Terry (about 9.5 kg, ♂) served for the experiments in question. Twice a day always at the same time (about 9 h. 30 and 14 h. 30) for two successive days the dogs received a subcutaneous injection of 20 cm^3 of a sterile solution of the disodium salt of the dicarboxylic acid to be investigated. The injection fluids were obtained by adding together the pure dicarboxylic acid and an aqueous solution of the theoretical amount of pure sodium carbonate and boiling until the carbon dioxide set free was completely removed. They always contained 0.0103 grammol. of dicarboxylic acid per 100 cm^3 and will thus always have been practically isotonic with the blood. They were feebly alkaline with respect to phenolphthalein; the p_H of the injection fluids prepared from the dicarboxylic acids included in this investigation (C_6, C_8, C_{10}, C_{13}) was not appreciably different, as both the first and the second dissociation constants of these acids differ fairly little from each other ²).

Thus in these experiments the effect of supplying equimolecular amounts of the various dicarboxylic acids was compared, whereas this was done with equal amounts by weight of the disodium salts in our previous work, hence with a number of dicarboxylic acid molecules which decreases rapidly on ascent of the homologous series; for example, the ratio of the number of molecules then amounts to 1 : 0.66 for adipic acid (C_6) and brassylic acid (C_{13}). The procedure now adopted seems to us to be definitely more correct.

The injections took place in the inner surface of the thigh or in the chest and with the C_6-, C_8- and C_{10}-acid without any phenomena worthy of mention. With the C_{13}-

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¹) P. E. VERKADE, J. VAN DER LEE, A. J. S. VAN ALPHEN and M. ELZAS, Proc. Royal Acad. Amsterdam, **40**, 746 (1937); Z. physiol. Chem. **250**, 47 (1937).

²) See, for example, G. L. VOERMAN, Rec. trav. chim. **23**, 265 (1904); R. GANE and C. K. INGOLD, Journ. chem. Soc. **1928**, 1598; **1931**, 2153.

acid a considerable oedematous swelling and red coloration of the skin rapidly appeared at the injection places. These symptoms usually disappeared in the course of a few days; once an abscess was formed, which on opening furnished a considerable amount of a haemorrhagic mucopurulent exudate in which no dicarboxylic acid could be detected. Precisely with a view to the occurrence of such symptoms it seemed to us advisible to extend the injections over not more than two days. On continuing them longer the danger would indeed become continually greater that the series of comparative experiments with different sodium salts would have to be broken off untimely.

It may be mentioned here in passing that much more violent symptoms occurred after injections of the disodium salt of the C_{16}-dicarboxylic acid, carried out in quite the same way. It seems that the toxic action of the higher dicarboxylic acids administered subcutaneously increases on ascent of the homologous series. It would be of interest to institute a careful investigation into this matter. Such work is indeed on our program.

During the different experiments with the same dog we attempted to get the latter to consume always twice a day at fixed times a meal which was the same in composition and weight (during the injection days 50 g dog biscuit + 50 g lean horse meat, afterwards only 50 g dog biscuit). Thanks also to the fairly small extent of the meals we generally succeeded in this. Such a regulation of the food intake appeared to us to be desirable for safety in these comparative experiments, since in man we have observed an influence of carbohydrate intake on the extent of the diaciduria occurring after the administration of triundecylin. This fact has already been mentioned several times; we hope to discuss it extensively in later publications. Water was always provided *ad lib.* for the test animals. A reasonable interval always occurred between successive experiments with the same dog, during which the animal resumed its normal life.

The urine was collected on the two injection days and several (at least 6) succeeding days. We have indicated elsewhere [3]) to a sufficient extent the way in which the urine was worked up and the isolation of the dicarboxylic acid excreted therein unchanged. We have not troubled ourselves this time with the lower dicarboxylic acids produced therefrom by β-oxidation.

§ 3. The table below gives a review of the results obtained. As a consequence of the small amount of adipic acid (1.20 g) injected on the one hand and the fairly large solubility of this acid in water on the other hand, we have unfortunately not succeeded in making an exact determination of the amount of adipic acid excreted in the urine. This was all the less possible as certainly only a fairly small (dog Polly) or small (dog Terry) percentage of the amount of acid injected proved to be excreted.

	Amount injected	Excreted unchanged	
		dog Polly	dog Terry
C_6 adipic acid	1.20 g	several %	a few %
C_8 suberic acid	1.43 g	about 66 %	about 57 %
C_{10} sebacic acid	1.66 g	„ 16 %	„ 16 %
C_{13} brassylic acid	2.01 g	none	none

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[3]) See P. E. VERKADE and J. VAN DER LEE, Z. physiol. Chem. **227**, 215 (1934); P. E. VERKADE, J. VAN DER LEE, A. J. S. VAN ALPHEN and M. ELZAS, Proc. Royal Acad. Amsterdam, **38**, 945 (1935).

On comparing these results with those of the experiments with sodium salts supplied orally [1]) the difference in the fate of adipic acid strikes one. In the first case this acid is catabolised much better than suberic acid and in the second case somewhat worse. Unfortunately we have not yet succeeded in finding a really satisfactory explanation for this contrast. We presume that an investigation of what happens to the sodium salts or the dicarboxylic acids first in the stomach and later in the intestine will be able to clear up this point.

For the rest the results of the two researches show a striking agreement. We are now again led to the conclusion that *the "combustibility" of these acids in the organism increases rapidly on ascent of the homologous series* and this time this conclusion is, from the nature of the experiments, really undeniable.

It may be mentioned here that the investigation of the sodium salts occurred in the order C_{10}, C_{13}, C_8, C_6 with the dog Polly and in the order C_{13}, C_{10}, C_8, C_6 with the dog Terry, so that there can be no question of the results being explained by the animals gradually "learning better" how to degrade the dicarboxylic acid.

At the time of the publication of communication VIII we were not aware of the results of the latest feeding experiments with sodium salts of dicarboxylic acids by K. BERNHARD and Miss M. ANDREAE [4]), nor of those of the injection experiments with monoethyl esters of dicarboxylic acids by K. BERNHARD [5]). Hence we now make the remark that it is incomprehensible to us how Miss ANDREAE [6]) can still express a doubt of the correctness of the conclusion from our feeding experiments then already published in a preliminary form [7]). Indeed, the results of the investigations from FLASCHENTRÄGER's laboratory just quoted, although dealing with different test subjects or dogs and not always obtained under comparable conditions, do nevertheless as a whole point in the same direction. We shall esteem it a favour if FLASCHENTRÄGER and his coworkers will repeat carefully both our feeding experiments and our injection experiments.

§ 4. We now come to the question of the cause of the increase of the "combustibility" of the dicarboxylic acids with ascent in the homologous series. It was already mentioned in communication VIII that by the term "combustibility" we merely mean to indicate in a very general way the possibility of being catabolised and that the "velocity of oxidation" of a dicarboxylic acid — already in itself a very complex concept — need in no wise be *a priori* regarded as the only factor which determines its "combustibility".

F. P. MAZZA [8]), using the manometric technique of Barcroft-Warburg, has investigated the velocity of oxidation of some normal saturated dicarboxylic acids (C_4, C_5, C_6, C_8, C_9, C_{10}, C_{18}) by surviving liver and kidney tissue from the rabbit and the guinea pig. We shall not discuss this work in detail here. MAZZA formulates a number of conclusions which fit well with our own results and with our ideas — one may refer here merely to the somewhat vague conclusion that "gli acidi a massa molecolare maggiore sono più facilmente ossidati"—and which would for that matter be thankfully accepted by us were it not that we are very doubtful whether these conclusions are really all completely justified by the experimental data which when all said and done are rather meagre. Caution is still to be advised all the more as some analogous experiments bij EDSON [9]) with the liver tissue of the rat have furnished a different picture on the whole of the oxidisability of the dicarboxylic acids (C_4, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{16}). A new and careful comparative investigation of the oxidisability of these dicarboxylic acids by surviving tissue is definitely desirable.

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[4]) Miss M. ANDREAE, Inaug.-Diss. Univ. Zürich, 1937; K. BERNHARD and Miss M. ANDREAE, Z. physiol. Chem. **245**, 103 (1937).

[5]) Z. physiol. Chem. **246**, 133 (1937)

[6]) Loc. cit., p. 14.

[7]) P. E. VERKADE and J. VAN DER LEE, Rec. trav. chim. **54**, 893 (1935).

[8]) Arch. sci. biol. (Ital.) **22**, no. 3 (1936).

In our opinion the desired explanation is not or at least not mainly to be sought in an increase of the "velocity of oxidation" of the dicarboxylic acids on ascent of the series. We sometimes tend rather to the supposition *that the dicarboxylic acids can be stored temporarily in the body with increasing ease on ascent of the series.* They will then be excreted unchanged in the urine to a decreasing extent and it will be possible for them to be gradually burned to an increasing extent. It may be regarded as a point in favour of this supposition that the amount of lower dicarboxylic acids excreted in the urine and formed by β-oxidation of the dicarboxylic acid administered also appears to become smaller if the latter is situated higher in the series. At any rate the amount of "difficultly combustible" dicarboxylic acids present in the urine in our feeding experiments with hexadecanedioic acid [1]) on dogs was found to be strikingly small. It may be remarked that if an increase of the "velocity of oxidation" of the dicarboxylic acids were of predominant importance, one would rather expect the opposite.

We believe that we can reject out of hand the possibility of a considerable increase in the threshold concentration of the dicarboxylic acids in the blood on ascent of the homologous series, by means of which the increase of the "combustibility" of these acids could also be explained.

§ 5. There is a parallelism between the diacidogenic action of the lower, simple, normal saturated triglycerides with the dog [10]) and the "combustibility" of their ω-oxidation products when supplied to it by injection. It is indeed understandable that there is a tendency to reduce the explanation of the first mentioned phenomenon to that of the latter one. In our opinion it is however a question whether that would be permissible, that is to say, whether the "combustibility" of the dicarboxylic acids is the only necessary element for the explanation of the diacidogenic action of the triglycerides. There is namely fairly certainly no question of any such parallelism with man. The variation of the diacidogenic action of the triglycerides on ascent of the homologous series is different in this case from that with the dog[11]), while the "combustibility" of the dicarboxylic acids, in so far as one can form an opinion at present[4]), shows qualitatively the same picture with both. The explanation of the first mentioned phenomenon would thus have to be different here, must be sought more deeply than in the case of the dog. This does not seem to us acceptable for the time being.

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[9]) Biochem. J. **30**, 1855 (1936).

[10]) P. E. VERKADE, J. VAN DER LEE and A. J. S. VAN ALPHEN, Z. physiol. Chem. **247**, 111 (1937).

We wish to thank the van 't Hoff Fund and the Hoogewerff Fund for financial assistance in the prosecution of these researches.

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[11]) P. E. VERKADE and J. VAN DER LEE, Biochem. J. 28, 31 (1934).

Palæontology. — *On the fossil human skull recently described and attributed to* Pithecanthropus erectus *by* G. H. R. VON KOENIGSWALD [1]). By EUG. DUBOIS.

(Communicated at the meeting of March 26, 1938.)

An air mail letter from Dr. VON KOENIGSWALD, dated Bandoeng (Java) 3rd September 1937, which came to hand 14th September, made me happy with the announcement of the discovery of a Pithecanthropus-skull-cap. Two months later I learned from his article *"Pithecanthropus received into the human family"*, in *The Illustrated London News* of Dec. 11, 1937, that Dr. VON KOENIGSWALD first on Sept. 13 visited the site — from which one of his native collectors had sent him that large fragment —, to ascertain what had happened to the rest of the skull, and collected twenty-nine smaller fragments belonging to the same skull, scattered about the surface. The description of the "skull-cap" in the letter, however, apparently not only refers to the large fragment, photographs of which (seen from the outer side and from the inner side), together with the twenty-nine small fragments, illustrate the article in the *Illustrated London News,* but to a more "preparated" (properly: composed and completed) state of the skull, approaching the final stage, represented by the photographs in VON KOENIGSWALD's recent communication. According to the latter (p. 185), he collected the small fragments already on the 13th of August.

Consequently, his letter of the 3rd of September refers to an earlier stage of "preparation", in the sense of reconstruction. It, moreover, gives evidence of indecision of opinion concerning the Trinil-femur. Therefore, and because those two photographs in *The Illustrated London News* refer to the original fragmentary state of the skull, especially showing the significant thickness of the cranial bones, and giving an idea of the natural shape of the brain, I think it conducive to a clear conception of the matter, to reproduce those two illustrations (by the annexed Plates I and II) and to quote from the letter literally:

„Ich glaube dass es mir inzwischen gelungen ist, die Zugehörigkeit des Unterkieferfragments zum Pithecanthropus zu beweisen: denn aus den gleichen Schichten ist dieser Tage ganz unerwartet auch ein Schädeldach zu Tage gekommen, das völlig mit Ihrem Funde von Trinil übereinstimmt!

Das neue Schädeldach ist leider recht zerbrochen, doch sind die wich-

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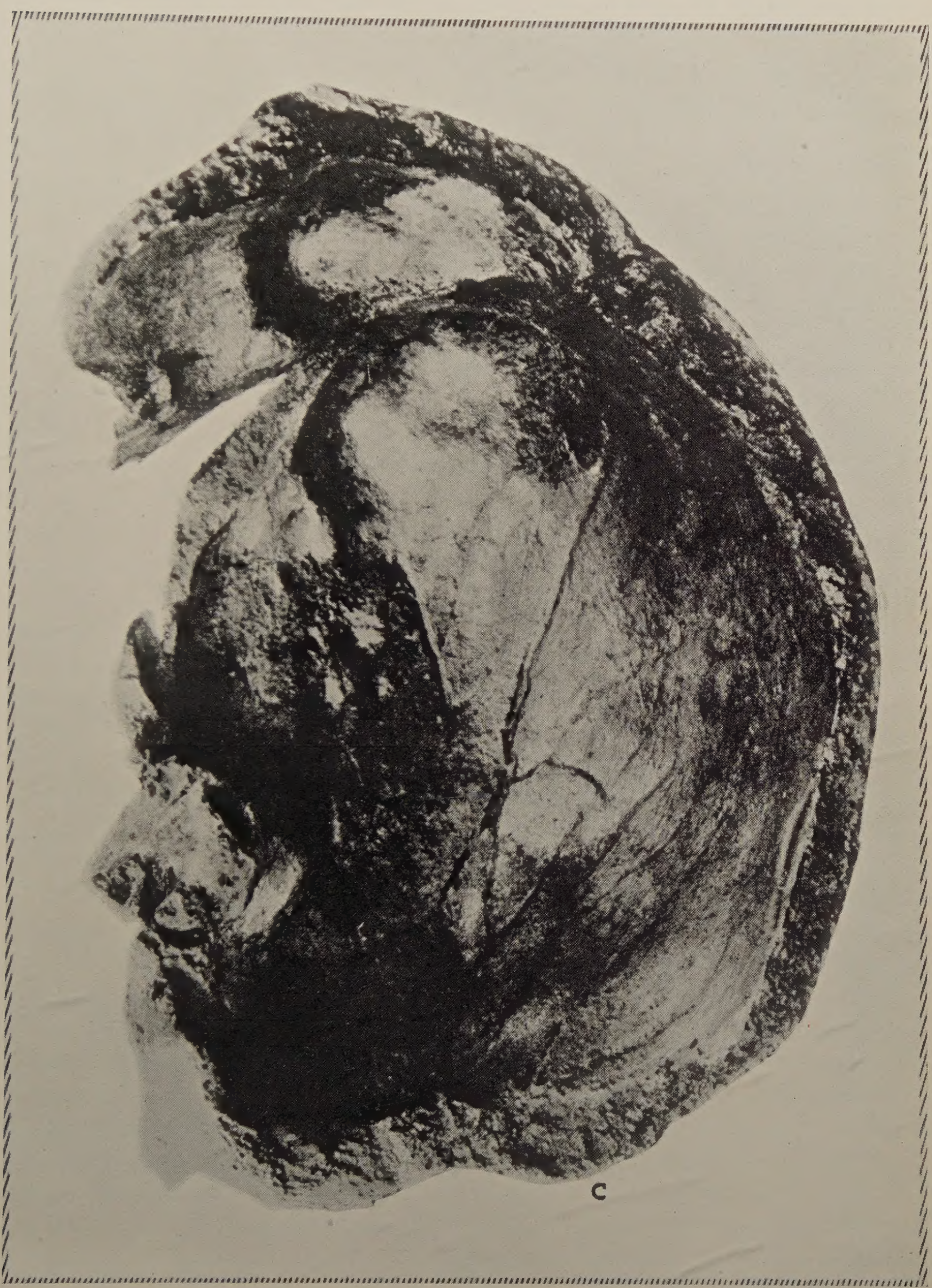
[1]) G. H. R. KOENIGSWALD, Ein neuer *Pithecanthropus*-Schädel. Proc. Royal Netherlands Acad. Amsterdam, **41**, 185—192 (1938). Communicated at the meeting of January 29, 1938.

EUG. DUBOIS: ON THE FOSSIL HUMAN SKULL RECENTLY DESCRIBED AND ATTRIBUTED TO PITHECANTHROPUS ERECTUS BY G. H. R. VON KOENIGSWALD.

PLATE I.



The chief part of the new fossil skull, seen from the inner side: large part of the left half of the cranial cavity, with occipital part of right half natural brain cast and enveloping bones. Reproduction of one of the illustrations to VON KOENIGSWALD's article "*Pithecanthropus received into the human family*", in *The Illustrated London News* of December 11, 1937. Enlarged to (approximately) natural size. Point *C* of the parietal bone corresponds to about the border of the coronal suture.